On the Josefson–Nissenzweig theorem for $C(K)$-spaces

Damian Sobota

Kurt Gödel Research Center for Mathematical Logic
University of Vienna

Joint work with Lyubomyr Zdomskyy.

The Josefson–Nissenzweig theorem

Theorem (Josefson–Nissenzweig)

For every infinite-dimensional Banach space X there exists a sequence of continuous functionals $\langle \varphi_n \colon\ n \in \omega \rangle$ on X such that $\|\varphi_n\| = 1$ for every $n \in \omega$ and $\varphi_n(x) \xrightarrow[n]{} 0$ for every $x \in X$.

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Proofs:

1. Josefson '75 (very intricate)
2. Nissenzweig '75 (intricate)
3. Hagler and Johnson '77 (legible, but the definition of φ_n's is non-constructive)
4. Behrends '94 (clear, quite elementary and constructive for $C(K)$)

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$\varphi \in C(K)^* \;\Rightarrow\;$ there exists a unique nice measure μ on K:

$$\varphi(f) = \int_K f \mathrm{d}\mu \quad , \; \forall f \in C(K)$$

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   - $\mu_n = \sum_{x \in A_n} \alpha_x^n \delta_x$ for some $\alpha_i^n \in \mathbb{R}$ and $A_n \in [D]^{<\omega}$,

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$$K = \beta\omega \colon \quad \mu_n = \delta_n \qquad \delta = 1$$

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1. $x_n \to x \in K \quad \Rightarrow \quad \langle \frac{1}{2}(\delta_{x_n} - \delta_x) \colon\ n \in \omega \rangle$ is JN.

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2. $\beta\omega$ does not have the JNP (Banakh–Kąkol–Śliwa '18)

More examples

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Corollary

For every K, the Alexandrov duplicate $AD(K)$ has the JNP, since

$$C_p(AD(K)) \simeq C_p(K \sqcup \alpha(|K|))$$

An example containing many copies of $\beta\omega$

Schachermayer's example:

$$\mathcal{B} = \{A \in \wp(\omega) \colon \exists N \; \forall n > N \colon \; 2n \in A \equiv 2n+1 \in A\}$$

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Remark: There are examples having similar properties but such that if $\langle \mu_n \colon\ n \in \omega \rangle$ is a JN-sequence, then $\lim_n \left|\operatorname{supp}\left(\mu_n\right)\right| = \infty$.

The Grothendieck property

A compact space K has ***the Grothendieck property*** (***the GP***) if for every sequence $\langle \mu_n \colon\ n \in \omega \rangle$ of measures on K we have:

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For K totally disconnected — equivalently:

$$\forall A \in Clopen(K)\colon\ \mu_n(A) \xrightarrow[n]{} 0 \ \Rightarrow\ \forall A \in Borel(K)\colon\ \mu_n(A) \xrightarrow[n]{} 0.$$

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1. K — extremely disconnected $\Rightarrow$ K has the Grothendieck property (Grothendieck '50s)
2. K has a convergent sequence $\Rightarrow$ K does not have the Grothendieck property.

The Grothendieck property and the JNP

1. K does not have the JNP if and only if $C_p(K)$ contains no complemented copy of $c_0 \subseteq \mathbb{R}^{\mathbb{N}}$ (Banakh–Kąkol–Śliwa '18).

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Corollary by the Closed Graph Theorem

If K has the Grothendieck property, then K does not have the Josefson–Nissenzweig property.

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Answer

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- $\mathcal{M} = \ell_1(K)$ — measures of countable support:
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Theorem

For a compact space K TFAE:

1. K has the JNP.
2. $\exists \langle \mu_n \in \ell_1(K) \colon \ n \in \omega \rangle \colon \ \|\mu_n\| = 1$, $\forall f \in C(K) \colon \ \mu_n(f) \xrightarrow[n]{} 0$

The Grothendieck property and the JNP

Let $\mathcal{M}$ be a subset of $C(K)^*$. A compact space K has ***the Grothendieck property for*** $\mathcal{M}$ (***the GP for*** $\mathcal{M}$) if for every sequence $\langle \mu_n \in \mathcal{M}\colon\ n \in \omega \rangle$ on K we have:
$\forall f \in C(K)\colon\ \mu_n(f) \xrightarrow[n]{} 0 \ \Rightarrow\ \forall f$ — Borel, bounded: $\mu_n(f) \xrightarrow[n]{} 0$

Examples:

- $\mathcal{M} = \ell_1(K)$ — measures of countable support:
  $\mu \in \ell_1(K) \ \equiv\ \mu = \sum_{n\in\omega} \alpha_n \delta_{x_n}\ ,\ x_n \in K\ ,\ \sum_{n\in\omega} |\alpha_n| < \infty$
- $\mathcal{M} = \operatorname{span}\delta(K)$ — measures of finite support

Theorem

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3. K does not have the GP for $\ell_1(K)$.
4. K does not have the GP for $\operatorname{span}\delta(K)$.

The Grothendieck property and the JNP

Plebanek's example

There exists a space K such that every separable closed subset $L \subseteq K$ has the Grothendieck property, but K does not have the Grothendieck property.

Corollary

There exists K without the GP and without the JNP.

In other words, there exists K without the GP but with the GP for $\ell_1(K)$.

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Question

Does there exist a separable space K without the GP but with the GP for $\ell_1(K)$? Hereditarily separable K?

Products and the JNP

Theorem (Khurana '78, Cembranos '84)

For every compact spaces K and L the product $K \times L$ does not have the Grothendieck property.

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Does $(\beta\omega)^2$ have the Grothendieck property for ℓ_1?

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$(\beta\omega)^2$ has the JNP, so it does not have the GP for $\ell_1((\beta\omega)^2)$.

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Minimal extensions

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A compact space K is ***obtained from a system of minimal extensions*** if K is the inverse limit of a system $\langle K_\alpha, \pi_\alpha^\beta \colon \alpha < \beta < \delta \rangle$ such that:

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- $K_{\alpha+1}$ is a minimal extension of K_α, i.e. there is a unique point $x_\alpha \in K_\alpha$ such that $\left|\left(\pi_\alpha^{\alpha+1}\right)^{-1}(x_\alpha)\right| = 2$ and $\left|\left(\pi_\alpha^{\alpha+1}\right)^{-1}(x)\right| = 1$ for every $x \neq x_\alpha$,

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Remark: Many consistent examples of Efimov spaces are obtained by minimal extensions, e.g.

Fedorchuk ($\Diamond$), Dow and Pichardo-Mendoza (CH), Dow and Shelah (MA+$\neg$CH) etc.

Minimal extensions and the JNP

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A JN-sequence on 2^ω

$$\forall n \in \omega, i \in 2, s \in 2^n: \;\; x_s^i := s^\frown iiiiiii\ldots$$

$$\mu_n = \frac{1}{2^{n+1}} \sum_{s \in 2^n} \left(\delta_{x_s^1} - \delta_{x_s^0}\right)$$

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Main Lemma

Let K be a compact totally disconnected space and $f\colon K \to 2^\omega$ be a continuous surjection such that $\lambda(f[U] \cap f[K \setminus U]) = 0$ for every clopen $U \subseteq K$.

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$$\mu_n = \frac{1}{2^{n+1}} \sum_{s \in 2^n} \left(\delta_{y_s^1} - \delta_{y_s^0}\right) \quad \text{defines a JN-sequence on } K.$$

Minimal extensions and Grothendieck property

Proposition (Borodulin-Nadzieja)

If K is a compact space obtained from a system of minimal extensions, then K does not have the Grothendieck property.

Corollary

If K is a compact space obtained from a system of minimal extensions of length at most $\mathfrak{c}$, then K does not have the Grothendieck property for $\ell_1(K)$.

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Question

What about systems of length $\geqslant \mathfrak{c}^+$?

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Question

What about systems of length $\geqslant \mathfrak{c}^+$?

Corollary from results of Borodulin-Nadzieja and Mercourakis

If K is a compact space obtained from a system of minimal extensions (of any length), then K has the JNP and hence does not have the Grothendieck property for $\ell_1(K)$.

The end

Thank you for the attention!