Almost disjoint families and C*-algebras

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1 Introduction

2 Projections of the Calkin algebra

3 Scattered $C^*$-algebras
   - $\Psi$-type $C^*$-algebras
   - Thin-tall $C^*$-algebras
Definition

A C*-algebra $\mathcal{A}$ is a structure $(\mathcal{A}, +, \cdot, *, ||||)$ such that

1. $(\mathcal{A}, +, \cdot, ||||)$ is a Banach algebra over $\mathbb{C}$,
2. $(a + b)^* = a^* + b^*$, $(\alpha a)^* = \overline{\alpha} a^*$, $(ab)^* = b^*a^*$,
3. $\|aa^*\| = \|a\|^2$ (the $C^*$-identity),

for every $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$.

Examples

- $M_n(\mathbb{C})$,
- $B(\mathcal{H})$ - The C*-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$,
- $K(\mathcal{H})$ - The ideal of all compact operators on $\mathcal{H}$,
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- For a locally compact Hausdorff space $X$, the space $C_0(X)$ with

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\begin{align*}
    f \cdot g(x) &= f(x)g(x), \\
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    \|f\| &= \sup\{f(x) : x \in X\},
\end{align*}
\]

is a commutative $C^*$-algebra.

Theorem (Gelfand)

Every commutative $C^*$-algebra is $*$-isomorphic to $C_0(X)$, for a locally compact Hausdorff space $X$. 
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Recall that an almost disjoint family $\mathcal{D} = \{D_\alpha : \alpha < \omega_1\} \subseteq \mathcal{P}(\mathbb{N})$ is called Luzin if for every $\alpha < \omega_1$ and $n \in \mathbb{N}$

$$\{\beta < \alpha : D_\alpha \cap D_\beta \subseteq n\}$$

is finite.

Facts

- There are Luzin families in ZFC.
- There are no separations of uncountable subfamilies i.e., given two disjoint uncountable $\mathcal{D}', \mathcal{D}'' \subseteq \mathcal{D}$ there is no $X \subseteq \mathbb{N}$ such that $A \subseteq^* X$ and $B \cap X =^* \emptyset$ for all $A \in \mathcal{D}'$ and $B \in \mathcal{D}''$. 
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Almost disjoint families in $C^*$-algebras

- $p \in A$ is a projection if $p^2 = p^* = p$.

Fix an orthonormal basis $\{e_x : x \in \mathbb{N}\}$ for $\ell_2$. For every $A \subseteq \mathbb{N}$ let $P_A$ denote the projection on the closed subspace spanned by $\{e_n : n \in A\}$.

$$P_A P_B \in \mathcal{K}(\ell_2) \iff A \cap B \in \text{Fin}.$$  

Definition (Wofsey)

For a Hilbert space $\mathcal{H}$, a family $\mathcal{P}$ of noncompact projections of $\mathcal{B}(\mathcal{H})$ is called almost orthogonal if the product of any two distinct elements is compact.

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• SOME APPLICATIONS
lifting projections of the Calkin algebra

\[ \mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H) := \text{The Calkin algebra.} \]

**Fact**

Every countable commuting family of projections of the Calkin algebra can be simultaneously lifted to a family of commuting projections in \( \mathcal{B}(H) \).

**Theorem (Anderson, 1979)**

Under CH there is an uncountable family \( \mathcal{P} \) of commuting projections in the Calkin algebra such that no uncountable \( \mathcal{P}_1 \subseteq \mathcal{P} \) can be simultaneously lifted to a family of commuting projections in \( \mathcal{B}(H) \).

**Theorem (Farah 2006, Bice-Koszmider 2016)**

There are \( \aleph_1 \) orthogonal projections in the Calkin algebra such that no uncountable subset can be simultaneously lifted to commuting projections in \( \mathcal{B}(H) \).
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Sketch of the proof

Fix a dense set of operators \( \{ K_n : n \in \mathbb{N} \} \subseteq \mathcal{K}(H) \) and \( 0 < \epsilon < 1/2 \).
Recursively construct a (Luzin-like) family of projections \( \{ P_\xi : \xi < \omega_1 \} \) in \( \mathcal{B}(H) \):

1. \( P_\xi P_\eta \in \mathcal{K}(H) \) for all \( \xi \neq \eta \),
2. for every \( \alpha < \omega_1 \) and for all \( n \), the set of all \( \beta < \alpha \) such that

\[
\| (P_\alpha - K_n)(P_\beta - K_n) - (P_\beta - K_n)(P_\alpha - K_n) \| < \epsilon
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Definition
A locally compact space $K$ is called **scattered** if every nonempty subset of $K$ has an isolated point. Equivalently every continuous image of $K$ has an isolated point.

Definition (Cantor-Bendixon Derivatives)
- $K^{(1)} = K'$ be the set of all non-isolated points of $K$,
- $K^{(\alpha+1)} = K^{(\alpha)'}$,
- $K^{(\gamma)} = \bigcap_{\alpha<\gamma} K^{(\alpha)}$, for limit ordinal $\gamma$.

- $K$ is scattered iff for for an ordinal $ht(K)$ (the **height** of $K$) such that $K^{ht(K)} = \emptyset$.
- The **width** of $K$ is the supremum of the cardinality of $K^{(\alpha)} \setminus K^{(\alpha+1)}$ for $\alpha < ht(K)$.
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Definition (Cantor-Bendixon Derivatives)
- $K^{(1)} = K'$ be the set of all non-isolated points of $K$,
- $K^{(\alpha+1)} = K^{(\alpha)'}$,
- $K^{(\gamma)} = \bigcap_{\alpha < \gamma} K^{(\alpha)}$, for limit ordinal $\gamma$.
- $K$ is scattered iff for for an ordinal $ht(K)$ (the height of $K$) such that $K^{ht(K)} = \emptyset$.
- The width of $K$ is the supremum of the cardinality of $K^{(\alpha)} \setminus K^{(\alpha+1)}$ for $\alpha < ht(K)$.
• Scattered $C^*$-algebras
Definition

isolated points $\iff$ minimal projections

A projection $p$ in $\mathcal{A}$ is called minimal if $pAp = \mathbb{C}p$.

- In $\mathcal{B}(H)$ minimal projections are projections onto one dimensional subspaces.
- In $\mathcal{C}(X)$ minimal projections correspond to the characteristic functions of isolated points of $X$.

Definition

A $C^*$-algebra $\mathcal{A}$ is called scattered if every nonzero subalgebra $B \subseteq \mathcal{A}$, has a minimal projection. Equivalently every non-zero $*$-homomorphic image of $\mathcal{A}$ has a minimal projection.

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For a C*-algebra $\mathcal{A}$, let $\mathcal{I}^{\text{At}}(\mathcal{A})$ denote the ∗-subalgebra of $\mathcal{A}$ generated by its minimal projections.

**Theorem**

Suppose that $\mathcal{A}$ is a C*-algebra.

1. $\mathcal{I}^{\text{At}}(\mathcal{A})$ is an ideal of $\mathcal{A}$,
2. $\mathcal{I}^{\text{At}}(\mathcal{A})$ is isomorphic to a subalgebra of $\mathcal{K}({\mathcal{H}})$ of compact operators on a Hilbert space $\mathcal{H}$,
3. $\mathcal{I}^{\text{At}}(\mathcal{A})$ contains all ideals of $\mathcal{A}$ which are isomorphic to a subalgebra of $\mathcal{K}({\mathcal{H}})$. 

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Almost disjoint families and C*-algebras 
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Suppose $\mathcal{A}$ is a scattered C*-algebra. We define the Cantor-Bendixson sequence $(\mathcal{I}_\alpha)_{\alpha \leq ht(\mathcal{A})}$ of ideals of $\mathcal{A}$ by induction:

- $\mathcal{I}_0 = \{0\}, \mathcal{I}_{ht(\mathcal{A})} = \mathcal{A},$
- $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha = \mathcal{I}^{At}(\mathcal{A}/\mathcal{I}_\alpha), \text{ for } \alpha < ht(\mathcal{A}),$
- If $\gamma$ is a limit ordinal let $\mathcal{I}_\gamma = \bigcup_{\alpha < \gamma} \mathcal{I}_\alpha.$
The Cantor-Bendixson composition series

Suppose $\mathcal{A}$ is a scattered $C^\ast$-algebra. We define the Cantor-Benndixson sequence $(\mathcal{I}_\alpha)_{\alpha \leq ht(\mathcal{A})}$ of ideals of $\mathcal{A}$ by induction:

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Examples

- $\mathcal{K}(H)$,
- $C_0(K)$, for $K$ scattered locally compact Hausdorff space,
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• \( \Psi \)-type \( C^* \)-algebras
Let $D = \{A_\xi : \xi < \kappa \}$ be an almost disjoint family of subsets of $\mathbb{N}$. The $\Psi(D)$ is the space $\mathbb{N} \cup D$, where all elements of $\mathbb{N}$ are isolated and the basic neighborhoods of $A_\xi \in D$ are of the form $\{A_\xi\} \cup A_\xi \setminus F$ for some finite set $F \subseteq \mathbb{N}$.

$\Psi(D)$ is a separable, scattered space of height two.

Faithfully represent $C_0(\Psi(D))$ in $B(\ell_2)$, by $\pi : C_0(\Psi(D)) \to B(\ell_2)$

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\pi(\chi_{\{n\}}) = \text{Proj } \text{span}\{e_n\},
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\pi(\chi_{A_\xi}) = \text{Proj } \text{span}\{e_n : n \in A_\xi\}.
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Let $P_\xi = \pi(\chi_{A_\xi})$. 
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\[ A_D = \pi[C(\Psi(D))] \] is the commutative C*-algebra generated by \( \{P_\xi : \xi < \kappa\} \) and \( c_0 \subseteq K(\ell_2) \).

For \( \xi \neq \eta \), we have \( P_\xi P_\eta \in K(\ell_2) \), since \( A_\xi \cap A_\eta \) is finite.

We have

\[ I_1 = c_0, \quad I_2/I_1 \cong c_0(\kappa), \quad I_2 = C_0(\Psi(D)). \]

Hence

\[ 0 \to c_0 \xrightarrow{\iota} C_0(\Psi(D)) \xrightarrow{\pi} c_0(\kappa) \to 0, \]

We would like to obtain a non-commutative version of this phenomena, i.e., a C*-algebra \( A \subseteq B(\ell_2) \) which contains \( K(\ell_2) \) as an (essential) ideal and satisfies

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We would like to obtain a non-commutative version of this phenomena, i.e, a C*-algebra $\mathcal{A} \subseteq B(\ell_2)$ which contains $\mathcal{K}(\ell_2)$ as an (essential) ideal and satisfies

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In order to build such $\mathcal{A}$, 

- Start with $\mathcal{A}_D$, 
  - add $\mathcal{K}(\ell_2)$, 
  - add partial isometries to $\{P_\xi : \xi < c\}$ sending the projection $P_\xi$ to $P_\eta$ for each pair $\xi, \eta < c$, i.e., elements $T_{\xi,\eta}$ of $B(\ell_2)$ such that $T_{\xi,\eta} T_{\xi,\eta}^* = P_\xi$ and $T_{\xi,\eta}^* T_{\xi,\eta} = P_\eta$.

$\mathcal{A}$ is simply subalgebra of $B(\ell_2)$ generated by $\mathcal{T} = \{T_{\xi,\eta} : \xi, \eta < \kappa\}$ and the compact operators $\mathcal{K}(\ell_2)$.

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Definition
We say $\mathcal{T} = \{ T_{\xi,\eta} : \xi, \eta < \kappa \} \subseteq B(\ell_2(\kappa))$ is a system of matrix units if and only if for every $\alpha, \beta, \xi, \eta < \kappa$,

1. $T_{\eta,\xi}^{*} = T_{\xi,\eta}$,
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Fact
A $C^*$-algebra generated by a system of matrix units $\{ T_{\xi,\eta} : \xi, \eta < \kappa \}$ is isomorphic to $K(\ell_2(\kappa))$. 
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We say $\mathcal{T} = \{T_{\xi,\eta} : \xi, \eta < \kappa\} \subseteq \mathcal{B}(\ell_2)$ is a system of almost matrix units if and only if for every $\alpha, \beta, \xi, \eta < \kappa$,

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$\mathcal{A}(\mathcal{T})$ is a scattered $C^*$-algebra of height 2.

$I_1 = \mathcal{K}(\ell_2), \quad I_2/I_1 \cong \mathcal{K}(\ell_2(\kappa)), \quad I_2 = \mathcal{A}(\mathcal{T}).$

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Theorem (Mrówka)

There is a maximal almost disjoint family $D$ of size $c$, such that $|\beta(\Psi(D)) \setminus \Psi(D)| = 1$.

Compactifications $\leftrightarrow$ Unitizations
One-point Compactification $\leftrightarrow$ The (minimal) unitization
Čech-Stone Compactification $\leftrightarrow$ Multiplier algebra

Theorem (G., Koszmider, 2016)

There is a system of almost matrix units $S$ of size $c$ such that $A(S)$ has the property that the multiplier algebra $M(A(S))$ of $A(S)$ is isomorphic to the (minimal) unitization of $A(S)$, i.e., $M(A(S))/A(S) \cong \mathbb{C}$.
**Theorem (Mrówka)**

*There is a maximal almost disjoint family $\mathcal{D}$ of size $\mathfrak{c}$, such that $|\beta(\Psi(\mathcal{D})) \setminus \Psi(\mathcal{D})| = 1.*

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*There is a system of almost matrix units $S$ of size $c$ such that $\mathcal{A}(S)$ has the property that the multiplier algebra $\mathcal{M}(\mathcal{A}(S))$ of $\mathcal{A}(S)$ is isomorphic to the (minimal) unitization of $\mathcal{A}(S)$, i.e., $\mathcal{M}(\mathcal{A}(S))/\mathcal{A}(S) \cong \mathbb{C}.$*
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There is a maximal almost disjoint family $D$ of size $c$, such that $|\beta(\psi(D)) \setminus \psi(D)| = 1$.

Compactifications $\leftrightarrow$ Unitizations
One-point Compactification $\leftrightarrow$ The (minimal) unitization
Čech-Stone Compactification $\leftrightarrow$ Multiplier algebra

Theorem (G., Koszmider, 2016)

There is a system of almost matrix units $S$ of size $c$ such that $A(S)$ has the property that the multiplier algebra $M(A(S))$ of $A(S)$ is isomorphic to the (minimal) unitization of $A(S)$, i.e., $M(A(S))/A(S) \cong \mathbb{C}$.
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Definition

A C*-algebra $\mathcal{A}$ is called **stable**, if $\mathcal{A} \otimes K(\ell_2) \cong \mathcal{A}$.

- For any infinite-dimensional Hilbert space $\mathcal{H}$, $K(\mathcal{H})$ is stable, since $K(\mathcal{H}) \otimes K(\ell_2) \cong K(\mathcal{H} \otimes \ell_2) \cong K(\mathcal{H})$.
- The Mrówka C*-algebra $\mathcal{A}(S)$ is not stable, since

$$B(\ell_2) \hookrightarrow M(\mathcal{A}(S)) \otimes B(\ell_2) \subseteq M(\mathcal{A}(S) \otimes K(\ell_2)).$$

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Extensions of $C^*$-algebras

Fact
It is well-known (Brown- Douglas-Fillmore) that for an extension

$$0 \to \mathcal{K}(\ell_2) \xrightarrow{\iota} A \xrightarrow{\pi} B \to 0$$

for separable $A$ and $B$, the $C^*$-algebra $A$ is stable if and only if $B$ is stable.

Not true for non-separable $C^*$-algebras, since

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Thin-tall $C^*$-algebras
A locally compact scattered space $K$ is called thin-tall if $ht(K) = \omega_1$, $wd(K) = \omega$.

- In 1978 Juhász and Weiss showed the existence of a compact thin-tall space.
- Simon and Weese were first to construct two nonisomorphc compact thin-tall spaces.
- Dow and Simon showed that in ZFC, there are $2^{\omega_1}$ (as many as possible) pairwise non-isomorphic compact thin-tall spaces.
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**Definition**

A $C^*$-algebra $\mathcal{A}$ is called **fully noncommutative thin-tall** if there is a sequence of ideals $(\mathcal{I}_\alpha)_{\alpha \leq \omega_1}$ of $\mathcal{A}$ is such that

1. $\mathcal{I}_0 = \{0\}$, $\mathcal{I}_{\omega_1} = \mathcal{A}$, $\mathcal{I}_\alpha \subseteq \mathcal{I}_{\alpha'}$ for $\alpha \leq \alpha' \leq \omega_1$,
2. $\mathcal{I}_\lambda = \bigcup_{\alpha < \lambda} \mathcal{I}_\alpha$ for all limit ordinals $\lambda \leq \omega_1$,
   For every $\alpha < \omega_1$
3. $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is an essential ideal of $\mathcal{A}/\mathcal{I}_\alpha$,
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**Theorem (G, Koszmider, 2017)**

There are at least two non-isomorphic fully noncommutative thin-tall $C^*$-algebra, a stable one and a non-stable one.
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**Theorem (G, Koszmider, 2017)**

*There are at least two non-isomorphic fully noncommutative thin-tall $C^*$-algebra, a stable one and a non-stable one.*
Construct a **Luzin-like** sequence \((A_\alpha)_{\alpha < \omega_1}\) of \(C^*\)-subalgebras of \(B(\ell_2)\) such that

- \(A_\alpha \cong \mathcal{K}(\ell_2)\)
- they are pairwise almost orthogonal, i.e., \(AA' =^\mathcal{K} 0\) for all \(A \in A_\alpha\), \(A' \in A_{\alpha'}\) for any \(\alpha < \alpha' < \omega_1\),
- Given any two uncountable \(X, Y \subseteq \omega_1\) and any choice of \(A_\alpha \in A_\alpha\) for \(\alpha \in X\) and \(B_\beta \in A_\beta\) for \(\beta \in Y\) there is no projection \(P \in B(\ell_2)\) satisfying
  \[PA_\alpha =^\mathcal{K} A_\alpha\] for all \(\alpha \in X\) and \(PB_\beta =^\mathcal{K} 0\) for all \(\beta \in Y\).
  
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The idea

Construct a Luzin-like sequence \((\mathcal{A}_\alpha)_{\alpha<\omega_1}\) of \(C^*\)-subalgebras of \(B(\ell_2)\) such that

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References:


2. S. Ghasemi, P. Koszmider; An extension of compact operators by compact operators with no nontrivial multipliers. Matharxiv.


Thank you