An Interplay between Ultrafilters and Boolean Topological Groups

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A group G is Boolean if $g^2 = e$ for any $g \in G$.

All Boolean groups are

1. Abelian
2. vector spaces over $\mathbb{Z}_2$
3. $\Rightarrow$ free (algebraically)

Any Boolean group with basis X is isomorphic to the set $[X]^{<\omega}$ of finite subsets of X under the operation $\triangle$ of symmetric difference: $A \triangle B = (A \cup B) \setminus (A \cap B)$

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$B_n(X) = \{\text{elements of length } n\}, \qquad B(X) = \bigcup_{n\in\omega} B_n(X)$

A topological space is said to be extremally disconnected if the closure of any open set in this space is open (or, equivalently, the closures of any two disjoint open sets are disjoint).

Problem (Arhangelskii, 1967)

Does there exist in ZFC a nondiscrete extremally disconnected topological group?

Problem (Protasov, 1994)

Does there exist in ZFC a countable nondiscrete topological group in which all discrete subsets are closed?

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Remark

Closed sets in a hereditarily normal extremally disconnected space cannot have precisely one common limit point.

Zelenyuk (2000): If there are no P-point ultrafilters, then all countable discrete sets in any extremally disconnected group are closed.

Theorem (Reznichenko + S., 2016)

1. *Any countable nondiscrete topological group with nonrapid filter of neighborhoods of the identity elements contains a discrete sequence with precisely one limit point.*
2. *If there are no rapid ultrafilters, then any countable nondiscrete Boolean topological group contains two disjoint discrete subsets for each of which zero is a unique limit point.*

Theorem (Reznichenko + S., 2016)

1. *Any countable nondiscrete topological group with nonrapid filter of neighborhoods of the identity elements contains a discrete sequence with precisely one limit point.*
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Corollary

1. *It is consistent with ZFC that any countable nondiscrete topological group contains a discrete sequence with precisely one limit point.*
2. *The nonexistence of a countable nondiscrete extremally disconnected group is consistent with ZFC.*

A filter $\mathscr{F}$ on ω is

P-point, or simply *P*-filter:

Given any partition $\omega = \sqcup_{i\in\omega} C_i$, $C_i \notin \mathscr{F}$, there exists an $A \in \mathscr{F}$ such that $|A \cap C_i| < \aleph_0$ for all $i \in \omega$;

Q-point, or *Q*-:

Given any partition $\omega = \sqcup_{i\in\omega} F_i$, where all F_i are finite, there exists an $A \in \mathscr{F}$ such that $|A \cap F_i| \leq 1$ for all $i \in \omega$;

selective, or Ramsey:

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rapid:

Given any partition $\omega = \sqcup_{i\in\omega} F_i$, where all F_i are finite, there exists an $A \in \mathscr{F}$ such that $|A \cap C_i| \leq n$ for all $i \in \omega$.

CH $\implies$ $\exists$ selective ultrafilters, $P \neq Q \neq$ selective $\neq P$

ZFC $\implies$ $\exists$ an ultrafilter which is neither a P-point nor a Q-point

Shelah: There is a model in which $\nexists$ P-point ultrafilters

Miller: In Laver's model $\nexists$ Q-points (but $\exists$ P-points)

Old problem: Does there exist a model in which there are no P-points and no Q-points?

The filter of neighborhoods of G is nonrapid if, given any sequence $(m_n)_{n\in\omega}$ of positive integers, there exist finite sets $F_n \subset \omega$, $n \in \omega$, such that each neighborhood U of e intersects some F_n in at least m_n points.

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If $S_n \subset G$ are closed, $S_{n+1} \subset S_n$, and $\bigcap S_n = \{e\}$, then the set

$$D = \bigcup_{n\in\mathbb{N}} \{a^{-1}b : a \neq b, a, b \in F_n, a^{-1}b \in S_n\}$$

is discrete, and either D is closed or e is its only limit point.

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Question

Is it true that a countable Boolean topological group with nonrapid neighborhood filter of zero is never extremally disconnected?

Question

Does there exist in ZFC a countable (Boolean) topological group containing no two disjoint discrete subsets for each of which zero is the only limit point?

Given a countable Boolean group B and any $A \subset G$ having a limit point with nonrapid filter of neighborhoods (in A), we construct a discrete set $D \subset A + A$ for which 0 is the only limit point (and has nonrapid filter of neighborhoods in $A + A$).

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We say that a set A in a Boolean group B is k-independent, $k \in \omega$, if $0 \notin A$ and there are no different $a_1, \ldots, a_k \in A$ for which $a_1 + \cdots + a_k = 0$. A set is independent if it is k-independent for all k.

Independent = linearly independent

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Theorem

If B is a countable Boolean topological group and all convergent (ultra)filters on B are nonrapid, then the following sets give rise to two discrete sets for each of which 0 is the only limit point:

- *any 4-independent set with at least two limit points;*
- *any 4- and 6-independent set with at least one limit point;*
- *any 3-independent set accumulating to zero.*

Corollary

If all convergent (ultra)filters on a countable extremally disconnected group are nonrapid, then any independent set in this group is closed and discrete.

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Remark

Any countable topological group in which the neighborhood filter of zero is a Q-filter contains a nondiscrete independent set.

Theorem (Sirota, Thümmel, Zelenyuk, S.)

An ultrafilter on a countable Boolean group is selective $\iff$ *it contains an independent set and the maximal group topology in which it converges is extremally disconnected.*

Theorem (Zelenyuk, 2000) ($\mathfrak{p} = \mathfrak{c}$)

There exists a nonselective P-point ultrafilter on the countable Boolean group such that the maximal group topology in which it converges is extremally disconnected.

Corollary′

If all convergent (ultra)filters on a countable extremally disconnected group are nonrapid, then no 3-independent set in this group accumulates to zero.

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Theorem

Suppose that $\mathscr{F}$ is a convergent uniform filter on a (countable or uncountable) Boolean group and $\mathscr{F}$ contains an independent set A. Then no 3-independent set in $A + A$ accumulates to zero $\iff$ $\mathscr{F}$ is 3-arrow.

Definition

Let κ be an infinite cardinal, and let $\mathscr{F}$ be a uniform filter on κ. Given any cardinal $\lambda \le \kappa$, $\mathscr{F}$ is a λ-arrow filter if, for any 2-coloring $c\colon [X]^2 \to \{0, 1\}$, there exists either a set $A \in \mathscr{F}$ such that $c([A]^2) = \{0\}$ or a set $S \subset X$ with $|S| \ge \lambda$ such that $c([S]^2) = \{1\}$.

- Any λ-arrow filter is an ultrafilter;
- An ultrafilter on ω is Ramsey $\iff$ it is ω-arrow;
- Any 3-arrow filter on ω can be mapped to a Ramsey ultrafilter;
- Any 3-arrow filter on any κ is not $(\omega, 2^\omega)$-regular;
- (CH) A 3-arrow filter on uncountable κ exists $\implies$ there is an inner model with a measurable cardinal.

Big Problem (Arhangelskii, 1967)

Does there exist in ZFC an uncountalbe nondiscrete extremally disconnected topological group?

THANK YOU