Cichoń's Maximum

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Outline

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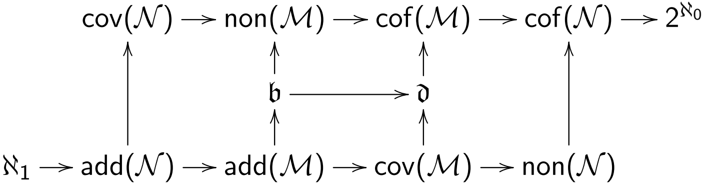
Cichoń's Diagram

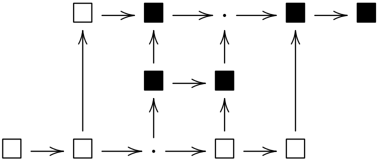
$\mathcal{M}$ = the ideal of meager subsets of $\mathbb{R}$.
$\mathcal{N}$ = the ideal of Lebesgue null sets of $\mathbb{R}$.



Are these cardinals different?

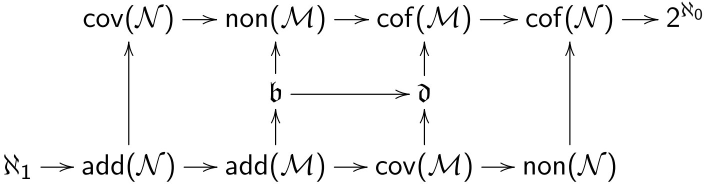
Examples

- CH $\Leftrightarrow$ all these cardinals are equal.
- MA $+ \neg$CH $\Rightarrow$ 2 values: $\aleph_1 < \mathrm{add}(\mathcal{N}) = 2^{\aleph_0}$.
- Many other consistency results for 2 values. e.g.



- Many consistency results for more than 2 values.

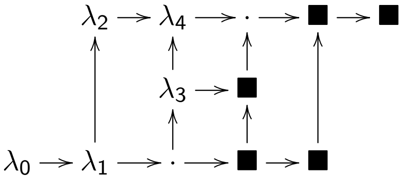
Outline



In ZFC:

$$\mathrm{add}(\mathcal{M}) = \min(\mathfrak{b}, \mathrm{cov}(\mathcal{M}))$$
$$\mathrm{cof}(\mathcal{M}) = \max(\mathrm{non}(\mathcal{M}), \mathfrak{d})$$

The left side



General strategy: E.g., to get $\mathrm{cov}(\mathcal{N}) \geq \lambda_2$, iterate (with finite support) for a long time, and make sure to take care of all "small" families F of measure zero sets by adding a random real over F. ("small" means: $< \lambda_2$.)
Hopefully, will not make $\mathrm{cov}(\mathcal{N}) > \lambda_2$.
For simplicity, we will today only consider $\mathrm{cov}(\mathcal{N})$ and $\mathfrak{b}$ on the left side, $\mathfrak{d}$ and $\mathrm{non}(\mathcal{N})$ on the right side.

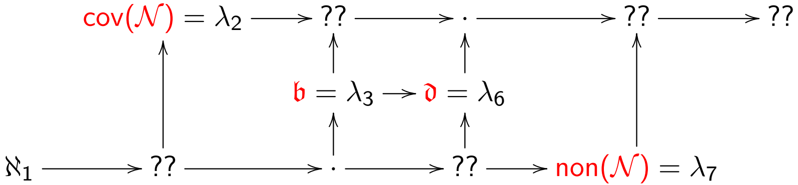
Main theorem

(G-Kellner-Shelah 2017, `arXiv:1708.03691`)

Starting from a universe with 4 strongly compact cardinals, we construct a universe in which 10 values $\aleph_1 = \lambda_0 < \cdots < \lambda_9 = 2^{\aleph_0}$ appear in Cichon's diagram:

$$
\begin{array}{ccccccccccc}
 & & \lambda_2 & \to & \lambda_4 & \to & \cdot & \to & \lambda_8 & \to & \lambda_9 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & \lambda_3 & \to & \lambda_6 & & & & \\
 & & & & \uparrow & & \uparrow & & & & \\
\lambda_0 & \to & \lambda_1 & \to & \cdot & \to & \lambda_5 & \to & \lambda_7 & &
\end{array}
$$

A fragment of the main theorem



How to make $\mathfrak{b}$ large, say: $\mathfrak{b} \geq \lambda$?

— Iterate a long time.
In each step add a real dominating some set of size $< \lambda$.
Use bookkeeping.
So every small set will be dominated.

How to make $\mathfrak{b}$ small, say: $\mathfrak{b} \leq \lambda$.

— Iterate λ steps (or at least with cofinality λ).
In each step add an unbounded real.
The generic reals will be an unbounded set.

How to ensure $\mathfrak{b} \geq \lambda_3$

A "standard" iteration is a FS (finite support) iteration $\vec{P} = (P_\alpha, Q_\alpha : \alpha < \delta)$ of ccc forcing notions together with a bookkeeping device $\vec{w} = (w_\alpha : \alpha < \delta)$, where:

- $\vec{w}$ is cofinal in $[\delta]^{<\lambda_3}$, and $\forall \alpha < \delta$: $w_\alpha \subseteq \alpha$
- Q_α adds a new generic c_α over $V^{\vec{P}\restriction w_\alpha}$.
  (A dominating real if we want to get $\mathfrak{b} \geq \lambda_3$)
- $V^{\vec{P}\restriction w_\alpha}$ is the model computed from $(c_\beta : \beta \in w_\alpha)$.
- To get $\mathfrak{b} \geq \lambda_3$ and $\mathrm{cov}(\mathcal{N}) \geq \lambda_2$, let $\delta = S^2 \mathbin{\dot\cup} S^3$, use cofinal families $\{w^2_\alpha : \alpha \in S^2\} \subseteq [\delta]^{<\lambda_2}$, $\{w^3_\alpha : \alpha \in S^3\} \subseteq [\delta]^{<\lambda_3}$, add random reals on S^2 and dominating reals on S^3.

WARNING: This is not trivial. Usually we want $=$, not $\geq$. Some work is needed to ensure $\mathfrak{b} \leq \lambda_3$, $\mathrm{cov}(\mathcal{N}) \leq \lambda_2$.
Use/Develop "preservation theorems".

Outline

Witnesses

- A *witness* for $\mathfrak{d} \leq \lambda$ is a family $(g_i : i < \lambda) \in (\omega^\omega)$ of functions $g_i \in \omega^\omega$ such that $\forall f \in \omega^\omega\ \exists i < \lambda : f \leq g_i$.
- A *witness* for $\mathfrak{b} \geq \lambda$ is a family $(f_i : i < \lambda) \in (\omega^\omega)$ of functions $f_i \in \omega^\omega$ such that $\forall g \in \omega^\omega\ \exists i < \lambda : f_i \not\leq g$.

Similar definitions can be made for the other characteristics. For example, a witness for $\mathrm{non}(\mathcal{M}) \leq \lambda$ is a family $(x_i : i < \lambda)$ of reals which is not meager (equivalently: for every code y of a meager Borel set M_y there is $i < \lambda$ such that $x_i \notin M_y$).
In the following slides we will only deal with $\mathfrak{b}$ and $\mathfrak{d}$; obvious (or at least: routine) modifications will yield appropriate definitions dealing with the other characteristics.

Strong witnesses (for $\mathfrak{b}$ small and $\mathfrak{d}$ large): linear witnesses

Recall: $\mathcal{F} = (f_i : i < \lambda)$ is a witness for $\mathfrak{b} \leq \lambda$ iff $\mathcal{F}$ is unbounded:

$$\forall g \in \omega^\omega : \exists i \; f_i \not\leq^* g$$

A linear λ-witness is a family $\mathcal{F} = (f_i : i < \lambda)$ of elements of ω^ω such that any g can only bound an initial segment of $\mathcal{F}$:

$$\forall g \in \omega^\omega : \forall^\infty i < \lambda : \; f_i \not\leq^* g$$

($\forall^\infty i < \lambda : \cdots$ means "eventually", i.e., $\exists i_0 \, \forall i \in (i_0, \lambda) : \cdots$)
$\mathrm{LCU}_{\mathfrak{b},\mathfrak{d}}(\lambda)$: "there is a linear witness of length λ".
FACT: $\mathrm{LCU}_{\mathfrak{b},\mathfrak{d}}(\lambda) \Rightarrow \mathfrak{b} \leq \lambda$, $\mathfrak{d} \geq \lambda$.
FACT: $\mathrm{LCU}(\lambda) \Leftrightarrow \mathrm{LCU}(cf(\lambda))$.
Similarly $\mathrm{LCU}_{\mathrm{cov}(\mathcal{N}),\mathrm{non}(\mathcal{N})}$.

Strong witnesses (for $\mathfrak{b}$ large and $\mathfrak{d}$ small): cone witnesses

Recall: $\mathcal{G} = (g_j : j < \lambda)$ is a witness for $\mathfrak{d} \le \lambda$ iff $\mathcal{G}$ dominates:

$$\forall f \in \omega^\omega : \exists j\ f \le^* g_j$$

Let λ, μ be regular uncountable. $\mathrm{COB}_{\mathfrak{b},\mathfrak{d}}(\lambda,\mu)$ means that there is a (λ,μ)-cone witness: a $<\lambda$-directed partial order $(S, \le)$ of size μ together with a sequence $(g_s : s \in S)$ of functions $g_s \in \omega^\omega$ such that

$$\forall f \in \omega^\omega\ \forall^\infty s \in S : f \le g_s$$

As above, $\forall^\infty s \in S$ means "eventually", i.e., $\exists s_0 \in S\ \forall s > s_0 \ldots$
FACT: $\mathrm{COB}_{\mathfrak{b},\mathfrak{d}}(\lambda,\mu) \Rightarrow \mathfrak{b} \ge \lambda, \mathfrak{d} \le \mu$.
We call the set $\{s \in S \mid s \ge s_0\}$ the "cone with tip s_0". If S is $<\lambda$-directed, then the cones generate a $<\lambda$-closed filter.

Strong witnesses, example 1 ($\mathfrak{b} \leq \lambda$)

Example

Let λ be regular uncountable. Let $(P_\alpha, Q_\alpha : \alpha < \lambda)$ be a finite support ccc iteration which adds (among other things) an unbounded real c_α at every step. Then P_λ (the FS limit of this iteration) forces that $\vec{c} = (c_\alpha : \alpha < \lambda)$ is a linear λ-witness. (Hence, P_λ forces that $\mathfrak{b} \leq \lambda$ and $\mathfrak{d} \geq \lambda$.)

Moreover: If $\lambda' > \lambda$, and we extend $(P_\alpha, Q_\alpha : \alpha < \lambda)$ to a longer iteration $(P_\alpha, Q_\alpha : \alpha < \lambda')$, and the forcings Q_α are "sufficiently nice", then $P_{\lambda'}$ will force that $(c_\alpha : \alpha < \lambda)$ remains a linear λ-witness, and also $(c_\alpha : \alpha < \lambda')$ becomes a linear λ'-witness. (So $P_{\lambda'}$ forces $\mathrm{LCU}_{\mathfrak{b},\mathfrak{d}}(\lambda)$ and $\mathrm{LCU}_{\mathfrak{b},\mathfrak{d}}(\lambda')$, so $\mathfrak{b} \leq \lambda$ and $\mathfrak{d} \geq \lambda'$.)

Strong witnesses, example 2 ($\mathfrak{b} \geq \lambda$)

Example

Let $(w_\alpha : \alpha < \delta)$ be a family of sets which is cofinal in $[\delta]^{<\lambda}$, with $w_\alpha \subseteq \alpha$ for all α.
Let $(P_\alpha, Q_\alpha : \alpha < \delta)$ be a "standard" finite support ccc iteration designed to make $\mathfrak{b} \geq \lambda$, based on $(w_\alpha : \alpha \in S) \subseteq [\delta]^{<\lambda}$, $S \subseteq \delta$ (each Q_α introduces a dominating c_α) over $V^{\vec{P} \restriction w_\alpha}$.
Then in V^{P_δ}, the sequence $(c_\alpha : \alpha \in S)$ is a $(\lambda, |S|)$-cone witness.
So we have $\mathrm{COB}_{\mathfrak{b},\mathfrak{d}}(\lambda, |S|)$, so $\mathfrak{b} \geq \lambda$, and $\mathfrak{d} \leq |S|$.
(We order S by $\alpha \sqsubseteq \beta \Leftrightarrow w_\alpha \subseteq w_\beta$. This partial order is clearly $<\lambda$-directed. Every P_δ-name of a real uses only few coordinates, hence will be in "almost all" $V^{P \restriction W_\alpha}$, therefore dominated by almost all c_α.)

Outline

Boolean ultrapowers (bups)

Let B be a κ-distributive Boolean algebra with the κ^+-cc.
A B-bup-name is an pair (A, f), where A is a maximal antichain in B and $f : A \to V$.
Essentially: A B-bup-name is the same as a name of an element of V, using B as a forcing notion. If τ and σ are B-names we write $[\tau = \sigma]$ for the Boolean value of the statement $\tau = \sigma$.
Let U be a $<\kappa$-complete ultrafilter on B. (So U meets all maximal antichains of B of size $< \kappa$, but in general not all those of size κ.)
Then U defines an equivalence relation $\tau \sim_U \sigma \Leftrightarrow [\tau = \sigma] \in U$.
The Boolean ultrapower $M = V^B/U$ is the set of all $\sim_U$-equivalence classes (after the Mostowski collapse). There is a natural embedding $j : V \to M$ using standard names.

Boolean ultrapowers, examples

Example

Let B be a complete Boolean algebra, and let $U \subseteq B$ be a V-generic ultrafilter.
Then every element of M is of the form $j(x)$, for some $x \in V$. So $M = V$, $j = id$.

Example

Let $B = \mathcal{P}(\kappa)$ be the powerset of κ. Then every antichain can be refined to the antichain $(\{\alpha\} : \alpha \in \kappa)$, so every B-bup-name is equivalent to a function $f : \kappa \to V$.
In this case M is the "traditional" ultrapower V^κ/U.

Boolean ultrapower embeddings

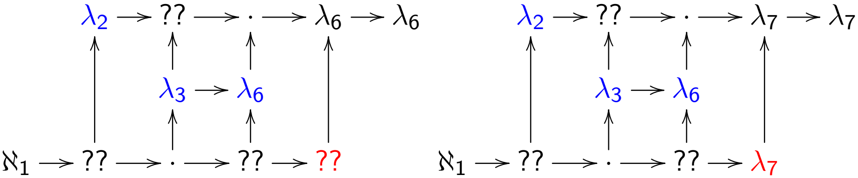
Assume GCH. Assume that κ is strongly compact. Then for every regular $\theta > \kappa$ there is an elementary embedding $j : V \to M$ with the following properties:

- $\kappa = cp(j)$
- $\theta \leq j(\kappa) \leq \theta^+$.
- (Every $x \in M$ is described by some (A, f) of size κ)
- If $(S, <)$ is $<\kappa^+$-directed in V, then $j''S$ is cofinal in $j(S)$.
- If $\lambda \neq \kappa$ is regular, then $cf(j(\lambda)) = \lambda$.
- If $\vec{P} = (P_\alpha, Q_\alpha : \alpha < \delta)$ is a FS ccc iteration, then $j(\vec{P})$ is a FS ccc iteration of length $j(\delta)$ not only in M, but also in V.

Note: M is $\leq\kappa$-closed. Contains all reals, even all names for reals.

REMARK: Moti Gitik suggested an extender ultrapower with a smaller large cardinal.

Outline



Assume P forces not only $\mathrm{cov}(\mathcal{N}) = \lambda_2$, $\mathfrak{b} = \lambda_3$, $\mathfrak{d} = \lambda_6$, but moreover:

$$\underset{\mathfrak{b}\leq\lambda_3}{\mathrm{LCU}_{\mathfrak{b},\mathfrak{d}}(\lambda_3)},\ \underset{\mathfrak{d}\geq\lambda_6}{\mathrm{LCU}_{\mathfrak{b},\mathfrak{d}}(\lambda_6)},\ \forall\lambda \in (\lambda_2, \lambda_3) : \underset{\mathrm{non}(\mathcal{N})\geq cf(\lambda)}{\mathrm{LCU}_{\mathrm{cov}(\mathcal{N}),\mathrm{non}(\mathcal{N})}(\lambda)}$$

$$\underset{\mathfrak{b}\geq\lambda_3,\ \mathfrak{d}\leq\lambda_6}{\mathrm{COB}_{\mathfrak{b},\mathfrak{d}}(\lambda_3, \lambda_6)},\ \underset{\mathrm{cov}(\mathcal{N})\geq\lambda_2,\ \mathrm{non}(\mathcal{N})\leq\lambda_6}{\mathrm{COB}_{\mathrm{cov}(\mathcal{N}),\mathrm{non}(\mathcal{N})}(\lambda_2, \lambda_6)}$$

Assume that κ is strongly compact, $\lambda_2 < \kappa < \lambda_3$. Let $j : V \to M$ be elementary with $cp(j) = \kappa$ and $cf(j(\kappa)) = \lambda_7$.
Then $j(P) \Vdash \mathrm{non}(\mathcal{N}) = \lambda_7$. (And the other cardinals stay.)

Proof sketch

$\lambda_2 < \kappa < \lambda_3$, $cf(j(\kappa)) = \lambda_7$.

- $\mathfrak{b}$ stays $\leq \lambda_3$:
  $P \Vdash \underbrace{\mathrm{LCU}_{\mathfrak{b},\mathfrak{d}}(\lambda_3)}_{\mathfrak{b}\leq\lambda_3}$, so $j(P)$ forces $\underbrace{\mathrm{LCU}_{\mathfrak{b},\mathfrak{d}}(j(\lambda_3))}_{\mathfrak{b}\leq j(\lambda_3)}$.
  But $\mathrm{LCU}_{\mathfrak{b},\mathfrak{d}}(\mu) \Leftrightarrow \mathrm{LCU}_{\mathfrak{b},\mathfrak{d}}(cf(\mu))$, so $j(P) \Vdash \mathfrak{b} \leq \lambda_3$.
- $\mathfrak{b}$ stays $\geq \lambda_3$:
  $P \Vdash \underbrace{\mathrm{COB}_{\mathfrak{b},\mathfrak{d}}(\lambda_3, \lambda_6)}_{\mathfrak{b}\geq\lambda_3}$, so $j(P) \Vdash \underbrace{\mathrm{COB}_{\mathfrak{b},\mathfrak{d}}(\lambda_3, \lambda_6)}_{\mathfrak{b}\geq\lambda_3}$.
  (Use $j''S$ as a witness! Isomorphic to S, hence same size λ_6.)
- $\mathrm{non}(\mathcal{N})$ becomes large:
  $P \Vdash \forall\lambda \in (\lambda_2, \lambda_3) : \dots$, in particular
  $P \Vdash \underbrace{\mathrm{LCU}_{\mathrm{cov}(\mathcal{N}),\mathrm{non}(\mathcal{N})}(\kappa)}_{\mathrm{non}(\mathcal{N})\geq cf(\kappa)}$, so $j(P) \Vdash \underbrace{\mathrm{LCU}_{\mathrm{cov}(\mathcal{N}),\mathrm{non}(\mathcal{N})}(\kappa)}_{\mathrm{non}(\mathcal{N})\geq cf(j(\kappa))}$.