Surgery and nonamalgability for Cohen reals

Miha E. Habič
(joint with Jonathan Verner)

Charles University, Prague

Winter School 2018, Hejnice

Amalgability

Fix a countable transitive model M and consider the family of extensions of M by a single Cohen real. Order this family by $\subseteq$. What can you say about this structure? In particular, we would like to study the existence of upper (and lower) bounds in this structure.

Definition

Say that a family $\mathcal{A}$ of Cohen reals over M is *amalgable* if there is a single Cohen real c over M such that $\mathcal{A} \subseteq M[c]$.

In other words, a family of Cohen reals is amalgable if the corresponding family of extensions has an upper bound in the above structure.

Some known results

Fact

Given any Cohen real, there is another Cohen real amalgable with it.

Some known results

Fact

Given any Cohen real, there is another Cohen real amalgable with it.

Fact (Woodin?/folklore)

There is a pair of nonamalgable Cohen reals.

Proof.

Enumerate the dense open subsets of the Cohen poset in M by $\langle D_n ; n < \omega \rangle$. Also, fix a very bad real z.

Some known results

Fact

Given any Cohen real, there is another Cohen real amalgable with it.

Fact (Woodin?/folklore)

There is a pair of nonamalgable Cohen reals.

Proof.

Enumerate the dense open subsets of the Cohen poset in M by $\langle D_n ; n < \omega \rangle$. Also, fix a very bad real z.

Some known results

Fact

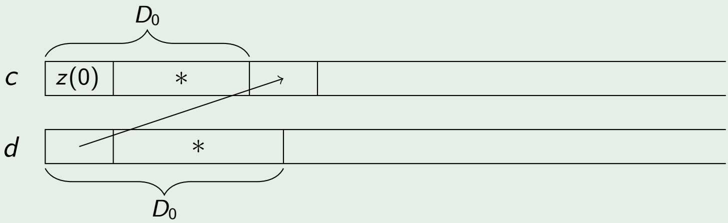
Given any Cohen real, there is another Cohen real amalgable with it.

Fact (Woodin?/folklore)

There is a pair of nonamalgable Cohen reals.

Proof.

Enumerate the dense open subsets of the Cohen poset in M by $\langle D_n; n < \omega \rangle$. Also, fix a very bad real z.



Some known results

Fact

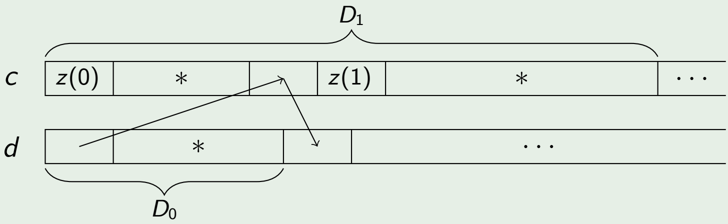
Given any Cohen real, there is another Cohen real amalgable with it.

Fact (Woodin?/folklore)

There is a pair of nonamalgable Cohen reals.

Proof.

Enumerate the dense open subsets of the Cohen poset in M by $\langle D_n; n < \omega \rangle$. Also, fix a very bad real z.



The main question

Question (Williams)

Given a number n and a family $A \subseteq [n]^2$, are there Cohen reals c_i for $i < n$ such that c_i and c_j are amalgable exactly if $\{i, j\} \in A$?

In other words, which patterns of pairwise (non)amalgability can you realize in the Cohen extensions?

It turns out that the main challenge is ensuring nonamalgability. As before, we need to find better and better ways of coding information into a Cohen real.

Surgery and covert sets

Definition

Let c be a Cohen real over M and $X \subseteq \omega$. Say that X is *c-covert* (over M) if, no matter how we modify c on the coordinates in X, the resulting real remains Cohen over M.
Say that X is *covert* (over M) if it is c-covert for every Cohen real c over M.

Note that it now becomes important how Cohen reals are presented.

Observation

If c is given as generic for the poset ${}^{<\omega}\omega$, then there are no c-covert sets X.

For this reason, we work from now on with the poset ${}^{<\omega}2$, i.e. binary sequences.

Examples of covert sets

- Every finite X is covert.
- No infinite $X \in M$ is c-covert for any c.
- Any subset of a c-covert set is c-covert. Consequently, any subset of a covert set is covert.

Theorem (Hamkins)

There are a Cohen real c over M and an infinite c-covert set $X \subseteq \omega$.

Theorem

Let c and d be mutually Cohen over M. Then d (as a subset of ω) is c-covert.

More covert sets

Definition (Blass/Solomon)

Let $I = \langle I_n; n < \omega \rangle$ and $J = \langle J_n; n < \omega \rangle$ be interval partitions of ω. Say that $I \leq^* J$ if all but finitely many pieces of J contain a piece of I.

Theorem

Suppose c is Cohen over M and $X \subseteq \omega$ gives an unbounded partition over $M[c]$. Then X is c-covert. Consequently, if X gives a dominating partition over M, then X is covert.

Question

Can a c-covert set be bounded over $M[c]$, or does the theorem characterize c-covertness?

Question

Is being dominating the same as being unbounded over every Cohen extension $M[c]$?

Back to amalgability

Theorem

Any Cohen real c is part of a nonamalgable pair.

Proof.

Let d be mutually Cohen with c and fix a very bad real z. Graft z onto d on the coordinates in c. The resulting real is Cohen over M and nonamalgable with c. $\square$

Back to amalgability

Theorem

Any Cohen real c is part of a nonamalgable pair.

Proof.

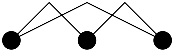
Let d be mutually Cohen with c and fix a very bad real z. Graft z onto d on the coordinates in c. The resulting real is Cohen over M and nonamalgable with c. □

So both of the amalgability patterns for 2 Cohen reals

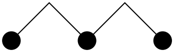
are realized (even while fixing one of the reals).

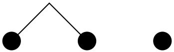
Combinations of 3 Cohen reals

Combinations of 3 Cohen reals

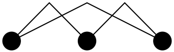


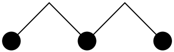
Three mutually Cohen reals



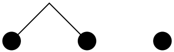


Combinations of 3 Cohen reals

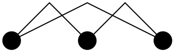


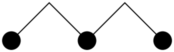


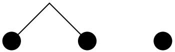
One Cohen real and two nonamalgable reals over it



Combinations of 3 Cohen reals

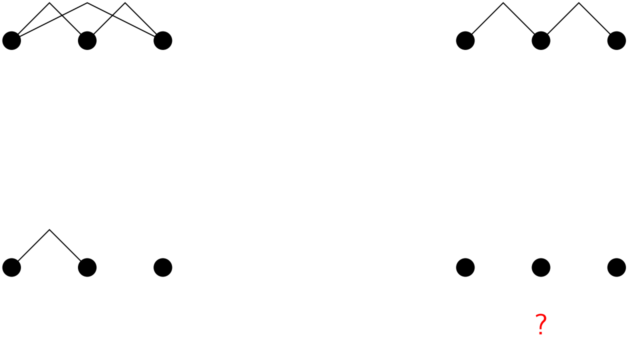






Two mutually Cohen reals;
graft two bad reals onto the third one

Combinations of 3 Cohen reals



More patterns

Observation

Suppose that $M \in M_1$ where M_1 is countable and M is countable in M_1. Then we can realize  *and therefore also $\bullet \;\bullet\; \bullet$ over M.*

Proof sketch.

Let $a \in M_1$ be Cohen over M. Let b be Cohen over M_1.

Work hard to get $c \in M_1$ Cohen over M which is nonamalgable with a and such that a remains unbounded over $M[c]$.

To get d, first get $d_0 \in M_1$ Cohen over $M[c]$ and nonamalgable with a; and second, get d_1 Cohen over M_1 and nonamalgable with b. The join $d = d_0 \oplus d_1$ is Cohen over $M[c]$ and nonamalgable with a and b. □

Small models

Definition

A countable transitive model M is *n-small* if there is an $\in$-chain of countable transitive models $M = M_0 \in M_1 \in \cdots \in M_{n-1}$ such that each model in the chain sees that each of its predecessors is countable.

Theorem

All combinations of 3 Cohen reals are realized over a 2-small model, and so is the moustache of length 4. Moreover, the moustache of length 5 is realized over a 3-small model.

Ongoing work

The constructions become more and more onerous, with a lot of bookkeeping. The tools shown do not seem to allow for inductive constructions; instead, each realization has to be built from scratch.

Very recently, with Joel David Hamkins, we have found a different construction that seems to embed any finite poset into the Cohen extensions, preserving the (non)existence of upper and lower bounds. This construction also builds the entire embedding at once.

Question

Is there an extension of embeddings phenomenon? In other words, given a poset and a realization of a subposet in the Cohen extensions, can this realization be extended to embed the whole poset?

Question

Which (infinite) countable patterns can be realized?

Thank you.