Large separated sets of unit vectors in Banach spaces of continuous functions

Ondřej Kurka

Charles University in Prague

based on a joint work with Marek Cúth and Benjamin Vejnar

Winter School in Abstract Analysis 2018
section Set Theory & Topology
Definition

We say that a set $A$ in a Banach space $X$ is $r$-separated (resp. $(r^+)$-separated) if

$$\|u - v\| \geq r \quad \text{(resp. } \|u - v\| > r)$$

for all distinct $u, v \in A$. 
**Definition**

We say that a set $A$ in a Banach space $X$ is *$r$-separated* (resp. *$(r+)$-separated*) if

$$
\|u - v\| \geq r \quad \text{(resp. } \|u - v\| > r)$$

for all distinct $u, v \in A$.

**Definition**

We say that a set $A$ in a Banach space $X$ is *$r$-equilateral* if

$$
\|u - v\| = r
$$

for all distinct $u, v \in A$. 
**Question A**

(i) If $X$ is a real infinite-dimensional Banach space, can we find a $(1 + \varepsilon)$-separated (resp. $(1+)$-separated) subset $A$ of the closed unit ball $B_X$ whose cardinality is $\text{dens}(X)$?

(ii) If not, how big separated set $A$ in $B_X$ can we find?
Question A

(i) If $X$ is a real infinite-dimensional Banach space, can we find a $(1 + \varepsilon)$-separated (resp. $(1+)$-separated) subset $A$ of the closed unit ball $B_X$ whose cardinality is $\text{dens}(X)$?
(ii) If not, how big separated set $A$ in $B_X$ can we find?
**Question A**

(i) If $X$ is a real infinite-dimensional Banach space, can we find a $(1 + \varepsilon)$-separated (resp. $(1+\varepsilon)$-separated) subset $A$ of the closed unit ball $B_X$ whose cardinality is $\text{dens}(X)$?

(ii) If not, how big separated set $A$ in $B_X$ can we find?

**Remark**

The closed unit ball of $c_0(\Gamma)$ does not contain an uncountable $(1 + \varepsilon)$-separated set.
Question A

(i) If $X$ is a real infinite-dimensional Banach space, can we find a $(1 + \varepsilon)$-separated (resp. $(1+)$-separated) subset $A$ of the closed unit ball $B_X$ whose cardinality is $\text{dens}(X)$?
(ii) If not, how big separated set $A$ in $B_X$ can we find?

Remark

The closed unit ball of $c_0(\Gamma)$ does not contain an uncountable $(1 + \varepsilon)$-separated set.

Remark

If $K$ is an infinite compact Hausdorff space, then the density $\text{dens}(C(K))$ equals to its weight $w(K)$. 

**Question A**

(i) If $X$ is a real infinite-dimensional Banach space, can we find a $(1 + \varepsilon)$-separated (resp. $(1+)$-separated) subset $A$ of the closed unit ball $B_X$ whose cardinality is $\text{dens}(X)$?
(ii) If not, how big separated set $A$ in $B_X$ can we find?

**Remark**

The closed unit ball of $c_0(\Gamma)$ does not contain an uncountable $(1 + \varepsilon)$-separated set.

**Question B**

(i) If $K$ is an infinite compact Hausdorff space, can we find a $(1 + \varepsilon)$-separated (resp. $(1+)$-separated) subset $A$ of the closed unit ball $B_{C(K)}$ of the space $C(K)$ whose cardinality is $w(K)$?
**Question A**

(i) If $X$ is a real infinite-dimensional Banach space, can we find a $(1 + \varepsilon)$-separated (resp. $(1+)$-separated) subset $A$ of the closed unit ball $B_X$ whose cardinality is $\text{dens}(X)$?

(ii) If not, how big separated set $A$ in $B_X$ can we find?

**Remark**

The closed unit ball of $c_0(\Gamma)$ does not contain an uncountable $(1 + \varepsilon)$-separated set.

**Question B**

(i) If $K$ is an infinite compact Hausdorff space, can we find a $(1 + \varepsilon)$-separated (resp. $(1+)$-separated) subset $A$ of the closed unit ball $B_{C(K)}$ of the space $C(K)$ whose cardinality is $w(K)$?

(ii) If not, how big separated set $A$ in $B_{C(K)}$ can we find?

T. Kania and T. Kochanek, *Uncountable sets of unit vectors that are separated by more than 1*, Studia Math. **232** (2016), 19–44.

P. Koszmider, *Uncountable equilateral sets in Banach spaces of the form $C(K)$*, accepted in Israel J. Math.
Remark

If $B_{C(K)}$ contains a $(1 + \varepsilon)$-separated set of cardinality $\kappa$, then it contains a 2-equilateral set of cardinality $\kappa$. 
The situation is clear if the density is countable:

**Theorem (Elton, Odell)**

*If* $X$ *is an infinite-dimensional Banach space, then there is* $\varepsilon > 0$ *such that* $B_X$ *contains an infinite* $(1 + \varepsilon)$-*separated set.*
The situation is clear if the density is countable:

**Theorem (Elton, Odell)**

*If* $X$ *is an infinite-dimensional Banach space, then there is* $\varepsilon > 0$ *such that* $B_X$ *contains an infinite* $(1 + \varepsilon)$-*separated set.*

**Corollary**

*If* $K$ *is an infinite compact Hausdorff space, then* $B_{C(K)}$ *contains an infinite 2-equilateral set.*
The situation is clear if the density is countable:

**Theorem (Elton, Odell)**

If $X$ is an infinite-dimensional Banach space, then there is $\varepsilon > 0$ such that $B_X$ contains an infinite $(1 + \varepsilon)$-separated set.

**Corollary**

If $K$ is an infinite compact Hausdorff space, then $B_{C(K)}$ contains an infinite 2-equilaterial set.

It is therefore possible to consider non-separable spaces only. In fact, we will focus on the $C(K)$ spaces only. So, from now, we assume that $K$ is a non-metrizable compact Hausdorff space.
The situation is clear if the density is countable:

**Theorem (Elton, Odell)**

*If X is an infinite-dimensional Banach space, then there is \( \varepsilon > 0 \) such that \( B_X \) contains an infinite \((1 + \varepsilon)\)-separated set.*

**Corollary**

*If K is an infinite compact Hausdorff space, then \( B_{C(K)} \) contains an infinite 2-equilateral set.*

It is therefore possible to consider non-separable spaces only.

In fact, we will focus on the \( C(K) \) spaces only. So, from now, we assume that \( K \) is a non-metrizable compact Hausdorff space.

The situation is not clear if the density is uncountable:

**Theorem (Koszmider)**

*It is undecidable in ZFC whether there exists an uncountable 2-equilateral set in \( B_{C(K)} \) for every such \( K \).*
Remark

It is not difficult to show that $B_{C(K)}$ contains a 1-separated set of cardinality $w(K)$. 
Remark
It is not difficult to show that $B_{C(K)}$ contains a 1-separated set of cardinality $w(K)$.

Question (Kania, Kochanek)
Does $B_{C(K)}$ always contain a (1+)-separated set of cardinality $w(K)$?
### Remark
It is not difficult to show that $B_{C(K)}$ contains a 1-separated set of cardinality $w(K)$.

### Question (Kania, Kochanek)
Does $B_{C(K)}$ always contain a $(1+)$-separated set of cardinality $w(K)$?

### Theorem (Kania, Kochanek)
*If $K$ is perfectly normal, then $B_{C(K)}$ contains a $(1+)$-separated set of cardinality $w(K)$.*
Remark
It is not difficult to show that $B_{C(K)}$ contains a 1-separated set of cardinality $w(K)$.

Question (Kania, Kochanek)
Does $B_{C(K)}$ always contain a $(1+)\text{-separated}$ set of cardinality $w(K)$?

Theorem (Kania, Kochanek)
If $K$ is perfectly normal, then $B_{C(K)}$ contains a $(1+)\text{-separated}$ set of cardinality $w(K)$.

Theorem 1
If $w(K)$ is at most continuum, then $B_{C(K)}$ contains a $(1+)\text{-separated}$ set of cardinality $w(K)$. 
Proposition 2

If $K$ contains a zero-dimensional compact subspace of the same weight as $K$, then $B_{C(K)}$ contains a 2-equilateral set of cardinality $w(K)$. 

Proof.
Let $L$ be such a subspace and let $f$ be a basis of $L$ consisting of clopen sets (clearly $w(L) = w(K)$).

Then the system $f$ given by $f(x) = \begin{cases} 1; & x \in U; \\ 1; & x \in L \cap U; \end{cases}$ forms a 2-equilateral set, and the Tietze theorem concludes the proof.
Proposition 2

If $K$ contains a zero-dimensional compact subspace of the same weight as $K$, then $B_{C(K)}$ contains a 2-equilateral set of cardinality $w(K)$.

Proof.

Let $L$ be such a subspace and let $\{U_\alpha\}_{\alpha < \kappa}$ be a basis of $L$ consisting of clopen sets (clearly $\kappa \geq w(L) = w(K)$). Then the system $\{f_\alpha\}_{\alpha < w(K)}$ given by

$$f_\alpha(x) = \begin{cases} 1, & x \in U_\alpha, \\ -1, & x \in L \setminus U_\alpha, \end{cases}$$

forms a 2-equilateral set, and the Tietze theorem concludes the proof.
Proposition 3

If $K$ contains a subset $A$ with $\text{dens}(A) \geq w(K)$, then $B_{C(K)}$ contains a 2-equilateral set of cardinality $w(K)$. 

Proof.

We inductively find points $x \in A$; $< w(K)$; such that $x = f(x) = g$. For each $< w(K)$, we pick a norm-one function $f$ such that $f(x) = 1$ and $f(x) = 1$ for $< g$. Then $f f$ is a 2-equilateral set.

Remark

A similar proof works if there is a point $x \in K$ with $(x; K) = w(K)$.

Corollary 4

If $K$ is a continuous image of a Valdivia compact space, then $B_{C(K)}$ contains a 2-equilateral set of cardinality $w(K)$. 

**Proposition 3**

If $K$ contains a subset $A$ with $\text{dens}(A) \geq w(K)$, then $B_{\mathcal{C}(K)}$ contains a 2-equilateral set of cardinality $w(K)$.

**Proof.**

We inductively find points $x_\alpha \in A$, $\alpha < w(K)$, such that $x_\alpha \notin \{x_\beta : \beta < \alpha\}$.

For each $\alpha < w(K)$, we pick a norm-one function $f_\alpha$ such that $f_\alpha(x_\alpha) = 1$ and $f_\alpha(x_\beta) = -1$ for $\beta < \alpha$.

Then $\{f_\alpha : \alpha < w(K)\}$ is a 2-equilateral set.
**Proposition 3**

If $K$ contains a subset $A$ with $\text{dens}(A) \geq w(K)$, then $B_{C(K)}$ contains a 2-equilateral set of cardinality $w(K)$.

**Proof.**

We inductively find points $x_\alpha \in A$, $\alpha < w(K)$, such that $x_\alpha \notin \{x_\beta : \beta < \alpha\}$.

For each $\alpha < w(K)$, we pick a norm-one function $f_\alpha$ such that $f_\alpha(x_\alpha) = 1$ and $f_\alpha(x_\beta) = -1$ for $\beta < \alpha$.

Then $\{f_\alpha : \alpha < w(K)\}$ is a 2-equilateral set.

**Remark**

A similar proof works if there is a point $x \in K$ with $\chi(x, K) \geq w(K)$.
**Proposition 3**

If $K$ contains a subset $A$ with $\text{dens}(A) \geq w(K)$, then $B_{\mathcal{C}(K)}$ contains a 2-equilateral set of cardinality $w(K)$.

**Proof.**

We inductively find points $x_\alpha \in A, \alpha < w(K)$, such that $x_\alpha \notin \{x_\beta : \beta < \alpha\}$.

For each $\alpha < w(K)$, we pick a norm-one function $f_\alpha$ such that $f_\alpha(x_\alpha) = 1$ and $f_\alpha(x_\beta) = -1$ for $\beta < \alpha$.

Then $\{f_\alpha : \alpha < w(K)\}$ is a 2-equilateral set.

**Remark**

A similar proof works if there is a point $x \in K$ with $\chi(x, K) \geq w(K)$.

**Corollary 4**

If $K$ is a continuous image of a Valdivia compact space, then $B_{\mathcal{C}(K)}$ contains a 2-equilateral set of cardinality $w(K)$. 
Proposition 5

If $K$ is a compact line (that is, a linearly ordered space with the order topology), then $B_{C(K)}$ contains a 2-equilateral set of cardinality $w(K)$. 
Theorem 6

$B_{C(K\times 2)}$ contains a 2-equilateral set of cardinality $w(K)$. 
**Theorem 6**

\( B_{C(K \times 2)} \) contains a 2-equilateral set of cardinality \( w(K) \).

**Proof.**

It is sufficient to find a \( \frac{3}{2} \)-separated set of cardinality \( w(K) \).

For \( f \in C(K \times 2) \) consider the following condition:

\[
\forall z \in K : \ |f(z, 0)| < \frac{1}{2} \implies f(z, 1) = -1. \tag{P}
\]

Take a maximal \( \frac{3}{2} \)-separated family \( \mathcal{F} \) (with respect to inclusion) of norm-one functions satisfying (P).

We claim that the cardinality of \( \mathcal{F} \) equals \( w(K) \). In order to get a contradiction, let us assume that \( \mathcal{F} \) does not separate the points of \( K \times \{0\} \). Thus, for some pair of distinct points \( x, y \in K \) and every \( g \in \mathcal{F} \), we have \( g(y, 0) = g(y, 0) \).

Now, consider any norm-one function \( f \in C(K \times 2) \) satisfying the condition (P) such that

\( f(y, 0) = -1 \) and \( f(x, 0) = f(x, 1) = 1 \). Such a function exists because we may pick any \( \tilde{f} \in B_{C(K)} \) with \( \tilde{f}(x) = 1 = -\tilde{f}(y) \) and take any continuous extension of a function defined on disjoint closed sets \( K \times \{0\}, \{(x, 1)\} \) and \( \tilde{f}^{-1}([\frac{-1}{2}, \frac{1}{2}]) \times \{1\} \) in the obvious way, that is, \( f(z, 0) = \tilde{f}(z) \) for every \( z \in K \), \( f(x, 1) = 1 \) and \( f(z, 1) = -1 \) for \( z \in \tilde{f}^{-1}([\frac{-1}{2}, \frac{1}{2}]) \).

Fix any \( g \in \mathcal{F} \).

If \( g(x, 0) = g(y, 0) \geq \frac{1}{2} \), then \( \|f - g\| \geq |1 - g(y, 0)| = 1 + g(y, 0) \geq \frac{3}{2} \).

If \( g(x, 0) = g(y, 0) \leq -\frac{1}{2} \), then \( \|f - g\| \geq |1 - g(x, 0)| = 1 - g(x, 0) \geq \frac{3}{2} \).

If \( |g(x, 0)| < \frac{1}{2} \), then since \( g \) satisfies (P) we have \( \|f - g\| \geq |f(x, 1) - g(x, 1)| = 1 - g(x, 1) = 2 \).

Therefore, we have \( \|f - g\| \geq \frac{3}{2} \) for any \( g \in \mathcal{F} \), which is a contradiction with the maximality of \( \mathcal{F} \). \( \square \)
Corollary 7

If $K$ is a compact convex set in a locally convex space, then $B_{C(K)}$ contains a 2-equilateral set of cardinality $w(K)$.
Corollary 7

If $K$ is a compact convex set in a locally convex space, then $B_{C(K)}$ contains a 2-equilateral set of cardinality $w(K)$.

Corollary 8

If $w(K) \geq (2^{<\kappa})^+$ for some cardinal $\kappa$, then $B_{C(K)}$ contains a 2-equilateral set of cardinality $\kappa$. 
Corollary 7

If $K$ is a compact convex set in a locally convex space, then $B_{C(K)}$ contains a 2-equilateral set of cardinality $w(K)$.

Corollary 8

If $w(K) \geq (2^{<\kappa})^+$ for some cardinal $\kappa$, then $B_{C(K)}$ contains a 2-equilateral set of cardinality $\kappa$.

Proof.

$(2^{<\kappa})^+ \rightarrow (\kappa)_2^2$ (Erdős, Rado).
Corollary 7

If $K$ is a compact convex set in a locally convex space, then $B_{C(K)}$ contains a 2-equilateral set of cardinality $w(K)$.

Corollary 8

If $w(K) \geq (2^{<\kappa})^+$ for some cardinal $\kappa$, then $B_{C(K)}$ contains a 2-equilateral set of cardinality $\kappa$.

Proof.

$(2^{<\kappa})^+ \rightarrow (\kappa)^2_2$ (Erdős, Rado).

Corollary 9 (GCH)

1. If $w(K)$ is a limit cardinal, then $B_{C(K)}$ contains a 2-equilateral set of cardinality $w(K)$.

2. If $w(K) = \kappa^+$ for an infinite cardinal $\kappa$, then $B_{C(K)}$ contains a 2-equilateral set of cardinality $\kappa$. 