Large separated sets of unit vectors in Banach spaces of continuous functions

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based on a joint work with Marek Cúth and Benjamin Vejnar

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section Set Theory & Topology

Definition

We say that a set A in a Banach space X is *r-separated (resp. (r+)-separated)* if

$$\|u - v\| \geq r \quad (\text{resp. } \|u - v\| > r)$$

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We say that a set A in a Banach space X is *r-equilateral* if

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(ii) If not, how big separated set A in $B_{C(K)}$ can we find?

- S. K. Mercourakis and G. Vassiliadis, *Equilateral sets in Banach spaces of the form $C(K)$*, Studia Math. **231** (2015), 241–255.
- T. Kania and T. Kochanek, *Uncountable sets of unit vectors that are separated by more than 1*, Studia Math. **232** (2016), 19–44.
- P. Koszmider, *Uncountable equilateral sets in Banach spaces of the form $C(K)$*, accepted in Israel J. Math.

Remark

If $B_{C(K)}$ contains a $(1+\varepsilon)$-separated set of cardinality κ, then it contains a 2-equilateral set of cardinality κ.

The situation is clear if the density is countable:

Theorem (Elton, Odell)

If X is an infinite-dimensional Banach space, then there is $\varepsilon > 0$ such that B_X contains an infinite $(1+\varepsilon)$-separated set.

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The situation is not clear if the density is uncountable:

Theorem (Koszmider)

It is undecidable in ZFC whether there exists an uncountable 2-equilateral set in $B_{C(K)}$ for every such K.

Remark

It is not difficult to show that $B_{C(K)}$ contains a 1-separated set of cardinality $w(K)$.

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Does $B_{C(K)}$ always contain a $(1+)$-separated set of cardinality $w(K)$?

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Theorem (Kania, Kochanek)

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Theorem 1

If $w(K)$ is at most continuum, then $B_{C(K)}$ contains a $(1+)$-separated set of cardinality $w(K)$.

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Proof.

Let L be such a subspace and let $\{U_\alpha\}_{\alpha<\kappa}$ be a basis of L consisting of clopen sets (clearly $\kappa \geq w(L) = w(K)$).
Then the system $\{f_\alpha\}_{\alpha<w(K)}$ given by

$$f_\alpha(x) = \begin{cases} 1, & x \in U_\alpha, \\ -1, & x \in L \setminus U_\alpha, \end{cases}$$

forms a 2-equilateral set, and the Tietze theorem concludes the proof. □

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We inductively find points $x_\alpha \in A, \alpha < w(K)$, such that $x_\alpha \notin \overline{\{x_\beta : \beta < \alpha\}}$.
For each $\alpha < w(K)$, we pick a norm-one function f_α such that $f_\alpha(x_\alpha) = 1$ and $f_\alpha(x_\beta) = -1$ for $\beta < \alpha$.
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Corollary 4

If K is a continuous image of a Valdivia compact space, then $B_{C(K)}$ contains a 2-equilateral set of cardinality $w(K)$.

Proposition 5

If K is a compact line (that is, a linearly ordered space with the order topology), then $B_{C(K)}$ contains a 2-equilateral set of cardinality $w(K)$.

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$B_{C(K\times 2)}$ *contains a 2-equilateral set of cardinality* $w(K)$.

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Proof.

It is sufficient to find a $\frac{3}{2}$-separated set of cardinality $w(K)$.
For $f \in C(K \times 2)$ consider the following condition:

$$\forall z \in K: \ |f(z,0)| < \tfrac{1}{2} \implies f(z,1) = -1. \tag{P}$$

Take a maximal $\frac{3}{2}$-separated family $\mathcal{F}$ (with respect to inclusion) of norm-one functions satisfying (P).
We *claim* that the cardinality of $\mathcal{F}$ equals $w(K)$. In order to get a contradiction, let us assume that $\mathcal{F}$ does not separate the points of $K \times \{0\}$. Thus, for some pair of distinct points $x, y \in K$ and every $g \in \mathcal{F}$, we have $g(x,0) = g(y,0)$. Now, consider any norm-one function $f \in C(K \times 2)$ satisfying the condition (P) such that $f(y,0) = -1$ and $f(x,0) = f(x,1) = 1$. Such a function exists because we may pick any $\tilde{f} \in B_{C(K)}$ with $\tilde{f}(x) = 1 = -\tilde{f}(y)$ and take any continuous extension of a function defined on disjoint closed sets $K \times \{0\}$, $\{(x,1)\}$ and $\tilde{f}^{-1}([-\frac{1}{2}, \frac{1}{2}]) \times \{1\}$ in the obvious way, that is, $f(z,0) = \tilde{f}(z)$ for every $z \in K$, $f(x,1) = 1$ and $f(z,1) = -1$ for $z \in \tilde{f}^{-1}([-\frac{1}{2}, \frac{1}{2}])$.
Fix any $g \in \mathcal{F}$.
If $g(x,0) = g(y,0) \geq \frac{1}{2}$, then $\|f - g\| \geq |-1 - g(y,0)| = 1 + g(y,0) \geq \frac{3}{2}$.
If $g(x,0) = g(y,0) \leq -\frac{1}{2}$, then $\|f - g\| \geq |1 - g(x,0)| = 1 - g(x,0) \geq \frac{3}{2}$.
If $|g(x,0)| < \frac{1}{2}$, then since g satisfies (P) we have $\|f - g\| \geq |f(x,1) - g(x,1)| = 1 - g(x,1) = 2$.
Therefore, we have $\|f - g\| \geq \frac{3}{2}$ for any $g \in \mathcal{F}$, which is a contradiction with the maximality of $\mathcal{F}$. □

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If $w(K) \geq (2^{<\kappa})^+$ for some cardinal κ, then $B_{C(K)}$ contains a 2-equilateral set of cardinality κ.

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$(2^{<\kappa})^+ \to (\kappa)^2_2$ (Erdős, Rado). □

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Corollary 9 (GCH)

1. *If $w(K)$ is a limit cardinal, then $B_{C(K)}$ contains a 2-equilateral set of cardinality $w(K)$.*
2. *If $w(K) = \kappa^+$ for an infinite cardinal κ, then $B_{C(K)}$ contains a 2-equilateral set of cardinality κ.*