Approximate Gowers spaces

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Winter School in Abstract Analysis,
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Hejnice, January 30, 2018
Infinite-dimensional Ramsey theory and the pigeonhole principle

Infinite-dimensional Ramsey theory is about coloring infinite sequences of objects.

**Theorem (Mathias-Silver)**

Let $\mathcal{X}$ be an analytic set of infinite subsets of $\mathbb{N}$. Then there exists $M \subseteq \mathbb{N}$ infinite such that:

- either for every infinite $A \subseteq M$, we have $A \in \mathcal{X}$;
- or for every infinite $A \subseteq M$, we have $A \notin \mathcal{X}$.

Here, the set $A$ can be viewed as an increasing sequence of integers.
Infinite-dimensional Ramsey theory and the pigeonhole principle

Let FIN denote the set of finite subsets of $\mathbb{N}$. Given a sequence $A_0 < A_1 < A_2 < \ldots$ of nonempty elements of FIN, a block-sequence is a sequence of the form $\bigcup_{i \in B_0} A_i$, $\bigcup_{i \in B_1} A_i$, $\bigcup_{i \in B_2} A_i$, $\ldots$ where $B_0 < B_1 < B_2 < \ldots$ is a sequence of nonempty elements of FIN.

**Theorem (Milliken)**

Let $\mathcal{X}$ be an analytic set of increasing sequences of nonempty elements FIN. Then there exists a sequence $A_0 < A_1 < A_2 < \ldots$ of nonempty elements of FIN such that:

- either every block-sequence of $(A_i)$ is in $\mathcal{X}$;
- or no block-sequence of $(A_i)$ is in $\mathcal{X}$.
Infinite-dimensional Ramsey theory and the pigeonhole principle

A *pigeonhole principle* is a one-dimensional Ramsey result, i.e. a Ramsey result where you color objects.
A *pigeonhole principle* is a one-dimensional Ramsey result, i.e., a Ramsey result where you color objects. Every infinite-dimensional Ramsey result has an associated pigeonhole principle, which is obtained by coloring sequences according to their first term.

The pigeonhole principle associated to Mathias-Silver's theorem is the following: for every coloring of the integers with two colors, there exists an infinite monochromatic subset.

The pigeonhole principle associated to Milliken's theorem is:

**Theorem (Hindman)**

For every coloring of the nonempty elements of $\mathbb{F}_2$, there exists a sequence $A_0 < A_1 < A_2 < \ldots$ of nonempty elements of $\mathbb{F}_2$ such that all the sets of the form $\bigcup_{i \in B} A_i$, for $B \in \mathbb{F}_2 \setminus \{\emptyset\}$, have the same color.

Can we still get something interesting without pigeonhole principle?
In infinite-dimensional Ramsey theory and the pigeonhole principle

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**Theorem (Hindman)**

For every coloring of the nonempty elements of FIN, there exists a sequence $A_0 < A_1 < A_2 < \ldots$ of nonempty elements of FIN such that all the sets of the form $\bigcup_{i \in B} A_i$, for $B \in \text{FIN} \setminus \{\emptyset\}$, have the same color.
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A pigeonhole principle is a one-dimensional Ramsey result, i.e. a Ramsey result where you color objects. Every infinite-dimensional Ramsey result has an associated pigeonhole principle, which is obtained by coloring sequences according to their first term.

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**Theorem (Hindman)**

*For every coloring of the nonempty elements of FIN, there exists a sequence \( A_0 < A_1 < A_2 < \ldots \) of nonempty elements of FIN such that all the sets of the form \( \bigcup_{i \in B} A_i \), for \( B \in \text{FIN} \setminus \{\emptyset\} \), have the same color.***

Can we still get something interesting without pigeonhole principle?
Gowers’ Ramsey-type theorems for Banach spaces

The first Ramsey-type result without pigeonhole principle is by Gowers, for Banach spaces.
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Fix $E$ a Banach space. Recall that a (normalized) sequence $(e_i)_{i \in \mathbb{N}}$ of $E$ is called a *Schauder basis* if for every $x \in E$, there exists a unique sequence $(x^i)_{i \in \mathbb{N}}$ of scalars such that $x = \sum_{i=0}^{\infty} x^i e_i$. 

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Approximate Gowers spaces
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A block-sequence of $(e_i)$ is a sequence $(x_i)_{i \in \mathbb{N}}$ of (normalized) vectors of $E$ with $\text{supp}(x_0) < \text{supp}(x_1) < \text{supp}(x_2) < \ldots$. A block-subspace is a (closed) subspace spanned by a block-sequence.
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We want a Ramsey result where we color block-sequences and where we find a monochromatic block-subspace.
Gowers’ Ramsey-type theorems for Banach spaces

The first Ramsey-type result without pigeonhole principle is by Gowers, for Banach spaces.

Fix $E$ a Banach space. Recall that a (normalized) sequence $(e_i)_{i \in \mathbb{N}}$ of $E$ is called a Schauder basis if for every $x \in E$, there exists a unique sequence $(x_i')_{i \in \mathbb{N}}$ of scalars such that $x = \sum_{i=0}^{\infty} x_i' e_i$.

A block-sequence of $(e_i)$ is a sequence $(x_i)_{i \in \mathbb{N}}$ of (normalized) vectors of $E$ with $\text{supp}(x_0) < \text{supp}(x_1) < \text{supp}(x_2) < \ldots$. A block-subspace is a (closed) subspace spanned by a block-sequence.

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**Notation**

1. For $A \subseteq S_E$ and $\delta > 0$, let $(A)_\delta = \{ y \in S_E \mid \exists x \in A \| x - y \| \leq \delta \}$.

2. For $\mathcal{X}$ a set of block-sequences and $\Delta$ a sequence of positive numbers, let $(\mathcal{X})_{\Delta}$ be the set of block-sequences $(y_n)$ for which there exists $(x_n) \in \mathcal{X}$ with $\forall n \in \mathbb{N} \| x_n - y_n \| \leq \Delta_n$.
Definition

Say that $E$ satisfies the **approximate pigeonhole principle** if for every $A \subseteq S_E$, for every (block) subspace $X \subseteq E$ and for every $\delta > 0$, there exists a (block) subspace $Y \subseteq X$ such that either $S_Y \subseteq (A)_\delta$, or $S_Y \subseteq (A^c)_\delta$.
Definition

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Theorem

$E$ satisfies the *approximate pigeonhole principle* iff it is $c_0$-saturated.

$\iff$ is by Gowers, $\Rightarrow$ comes from a combination of a result by Milman, and another by Odell and Schlumprecht.
Theorem (Gowers’ Ramsey-type theorem for $c_0$)

Let $E$ be a $c_0$-saturated Banach space with a Schauder basis $(e_i)$. Let $\Delta$ be a sequence of positive numbers, and $\mathcal{X}$ be an analytic set of block-sequences. Then there exists a block-subspace $X$ such that:

- either no block-sequence of $X$ is in $\mathcal{X}$;
- or every block-sequence of $X$ is in $\mathcal{X}^\Delta$. 

Gowers’ Ramsey-type theorems for Banach spaces

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Approximate Gowers spaces
To remedy to the lack of pigeonhole principle, we introduce Gowers’ game:

**Definition**

Let $E$ be a Banach space with a Schauder basis $(e_i)$, let $X$ be a block-subspace, and let $\mathcal{X}$ be a set of block-sequences of $(e_i)$. Gowers’ game $G_X(\mathcal{X})$ is defined as follows:

$$
\begin{array}{cccc}
& Y_0 & Y_1 & \ldots \\
I & & & \\
II & y_0 \in Y_0 & y_1 \in Y_1 & \ldots \\
\end{array}
$$

where the $Y_i$’s are block-subspaces of $X$, and the $y_i$’s are normalized vectors. Player II wins the game iff $(y_i)_{i \in \mathbb{N}}$ is a block-sequence that belongs to $\mathcal{X}$.
Theorem (Gowers’ Ramsey-type theorem)

Let $E$ be a Banach space with a Schauder basis $(e_i)$. Let $\Delta$ be a sequence of positive numbers, and $\mathcal{X}$ be an analytic set of block-sequences. Then there exists a block-subspace $X$ such that:

- either no block-sequence of $X$ is in $\mathcal{X}$;
- or $\Pi$ has a winning strategy in $G_X((\mathcal{X})_\Delta)$.
**Theorem (Gowers’ Ramsey-type theorem)**

Let $E$ be a Banach space with a Schauder basis $(e_i)$. Let $\Delta$ be a sequence of positive numbers, and $\mathcal{X}$ be an analytic set of block-sequences. Then there exists a block-subspace $X$ such that:

- either no block-sequence of $X$ is in $\mathcal{X}$;
- or $\mathbb{II}$ has a winning strategy in $G_X((\mathcal{X})_\Delta)$.

It turns out that this result has nothing to do with Banach spaces.
Let $P$ be a set (the set of subspaces) and $\leq$ and $\leq^*$ be two quasi-orderings on $P$, satisfying:

1. for every $p, q \in P$, if $p \leq q$, then $p \leq^* q$;
2. for every $p, q \in P$, if $p \leq^* q$, then there exists $r \in P$ such that $r \leq p$, $r \leq q$ and $p \leq^* r$;
3. for every $\leq$-decreasing sequence $(p_i)_{i \in \mathbb{N}}$ of elements of $P$, there exists $p^* \in P$ such that for all $i \in \mathbb{N}$, we have $p^* \leq p_i$;

Write $p \lessapprox q$ for $p \leq q$ and $q \leq^* p$. 

Let $(X, d)$ be Polish metric space (the set of points) and $\bowtie \subseteq X \times P$ a binary relation, satisfying:

4. for every $p \in P$, there exists $x \in X$ such that $x \bowtie p$;
5. for every $x \in X$ and every $p, q \in P$, if $x \bowtie p$ and $p \leq q$, then $x \bowtie q$.

The sextuple $G = (P, X, d, \leq, \leq^*, \bowtie)$ is called an approximate Gowers space.
Approximate Gowers spaces

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3. for every $\leq$-decreasing sequence $(p_i)_{i \in \mathbb{N}}$ of elements of $P$, there exists $p^* \in P$ such that for all $i \in \mathbb{N}$, we have $p^* \leq p_i$;

Write $p \preceq q$ for $p \leq q$ and $q \leq^* p$.

Let $(X, d)$ be Polish metric space (the set of points) and $\triangleleft \subseteq X \times P$ a binary relation, satisfying:

4. for every $p \in P$, there exists $x \in X$ such that $x \triangleleft p$.
5. for every $x \in X$ and every $p, q \in P$, if $x \triangleleft p$ and $p \leq q$, then $x \triangleleft q$.

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The sextuple $G = (P, X, d, \leq, \leq^*, \triangleleft)$ is called an *approximate Gowers space*. 
The formalism of Gowers spaces

Two examples

1. The Mathias-Silver space:
   - $X = \mathbb{N}$, and $\forall x, y \in \mathbb{N} \ (x \neq y \Rightarrow d(x, y) = 1)$;
   - $P$ is the set of infinite subsets of $\mathbb{N}$;
   - $\leq$ is the inclusion;
   - $\leq^*$ is the inclusion-by-finite;
   - $\triangleleft$ is the membership relation.

2. For $E$ a Banach space with a Schauder basis $(e_i)$, the standard Gowers space associated to $E$:
   - $X = S_E$ and $d$ is the usual distance;
   - $P$ is the set of block-subspaces of $E$;
   - $\leq$ is the inclusion;
   - $\leq^*$ is the inclusion up to finite dimension ($F \leq^* G$ iff $F \cap G$ has finite codimension in $F$);
   - $\triangleleft$ is the membership relation.
Approximate Gowers spaces

### Notation

1. For $A \subseteq X$ and $\delta > 0$, let $(A)_\delta = \{ y \in X | \exists x \in A \ d(x, y) \leq \delta \}$.

2. For $\mathcal{X} \subseteq X^\omega$ and $\Delta$ a sequence of positive numbers, let $(\mathcal{X})_\Delta = \{ (y_n) \in X^\omega | \exists (x_n) \in \mathcal{X} \ \forall n \in \mathbb{N} \ d(x_n, y_n) \leq \Delta_n \}$.
Notation

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   $(\mathcal{X})_{\Delta} = \{(y_n) \in X^\omega \mid \exists (x_n) \in \mathcal{X} \ \forall n \in \mathbb{N} \ d(x_n, y_n) \leq \Delta_n\}$.

Definition

Say that $G$ satisfies the pigeonhole principle if for every $A \subseteq X$, for every $p \in P$ and for every $\delta > 0$, there exists $q \leq p$ such that either $q \subseteq (A)_\delta$, or $q \subseteq (A^c)_\delta$. (Here, $q$ is identified with $\{x \in X \mid x \triangleleft q\}$.)
**Approximate Gowers spaces**

### Notation

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### Definition

Say that $G$ satisfies the *pigeonhole principle* if for every $A \subseteq X$, for every $p \in P$ and for every $\delta > 0$, there exists $q \leq p$ such that either $q \subseteq (A)_{\delta}$, or $q \subseteq (A^c)_{\delta}$. (Here, $q$ is identified with $\{x \in X \mid x \triangleleft q\}$.)

- The Mathias-Silver space satisfies the pigeonhole principle;
- The pigeonhole principle for the standard Gowers space associated to $E$ is precisely the approximate pigeonhole principle defined some slides ago.
Approximate Gowers spaces

Definition

Given $p \in P$ and $\mathcal{X} \subseteq X^\mathbb{N}$, Gowers’ game $G_p(\mathcal{X})$ is defined as follows:

Player I

\begin{align*}
& p_0 \leq p \\
& p_1 \leq p \\
& \ldots
\end{align*}

Player II

\begin{align*}
& x_0 \triangleleft p_0 \\
& x_1 \triangleleft p_1 \\
& \ldots
\end{align*}

Player II wins the game iff $(x_i)_{i \in \mathbb{N}} \in \mathcal{X}$.
We now add some structure to compensate for the lack of ordering on $X$.

Consider a family $\mathcal{K}$ of compact subsets of $X$ and a binary operation $\oplus$ on $\mathcal{K}$, associative and commutative, satisfying the following conditions:

- $\forall K_1, K_2 \in \mathcal{K}, K_1 \cup K_2 \subseteq K_1 \oplus K_2$;
- For all $p \in P$ and all $K_1, K_2 \in \mathcal{K}$, if $K_1 \triangleleft p$ and $K_2 \triangleleft p$, then $K_1 \oplus K_2 \triangleleft p$.

(Here, $K \triangleleft p$ means that $\forall x \in K x \triangleleft p$.)
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- For all $p \in P$ and all $K_1, K_2 \in \mathcal{K}$, if $K_1 \vartriangleleft p$ and $K_2 \vartriangleleft p$, then $K_1 \oplus K_2 \vartriangleleft p$.

(Here, $K \vartriangleleft p$ means that $\forall x \in K x \vartriangleleft p$.)

**Definition**

Given $p \in P$ and $\mathcal{X} \subseteq X^\mathbb{N}$, the **strong asymptotic game** $SF_p(\mathcal{X})$ is defined as follows:

I $p_0 \preceq p$ $p_1 \preceq p$ ... 
II $K_0 \vartriangleleft p_0$ $K_1 \vartriangleleft p_1$ ... 

where $K_i$’s are elements of $\mathcal{K}$. Player I wins the game iff for every sequence $A_0 < A_1 < \ldots$ of subsets of $\mathbb{N}$, we have 

$\bigoplus_{i \in A_0} K_i \times \bigoplus_{i \in A_1} K_i \times \cdots \subseteq \mathcal{X}$. 

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Theorem (dR.)

Let $p \in P$, $\Delta$ a sequence of positive numbers, and $\mathcal{X} \subseteq X^\omega$ analytic. Then there exists a $q \leq p$ such that:

- either I has a winning strategy in $SF_q(\mathcal{X}^c)$;
- or II has a winning strategy in $G_q((\mathcal{X})_\Delta)$. 

Moreover, if $G$ satisfies the pigeonhole principle, the second conclusion can be replaced with the following (stronger) one:

I has a winning strategy in $SF_q((\mathcal{X})_\Delta)$. 

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Approximate Gowers spaces
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Moreover, if $G$ satisfies the pigeonhole principle, the second conclusion can be replaced with the following (stronger) one:

- I has a winning strategy in $SF_q((\mathcal{X})_\Delta)$.
To get Mathias-Silver’s theorem, take for $\mathcal{K}$ the set of finite subsets of $\mathbb{N}$, and for $\oplus$ the union. Then, in $SF_q(\mathcal{X}^c)$, $\mathbb{I}$ just has to play a increasing sequence $\{n_0\}, \{n_1\}, \{n_2\}, \ldots$ of singletons, so any subsequence will be in $\mathcal{X}^c$.
Approximate Gowers spaces

To get Mathias-Silver’s theorem, take for $\mathcal{K}$ the set of finite subsets of $\mathbb{N}$, and for $\oplus$ the union. Then, in $SF_q(\mathcal{X}^c)$, $II$ just has to play an increasing sequence $\{n_0\}, \{n_1\}, \{n_2\}, \ldots$ of singletons, so any subsequence will be in $\mathcal{X}^c$.

To get Gowers’ theorem, take for $\mathcal{K}$ the set of $S_F$’s, where $F \subseteq E$ is a finite-dimensional block-subspace, where $S_F \oplus S_G = S_{F+G}$. Then in $SF_q(\mathcal{X}^c)$, $II$ just has to play a sequence $K_0 = \{x_{i_0}, -x_{i_0}\}, K_1 = \{x_{i_1}, -x_{i_1}\}, \ldots$, where $i_0 < i_1 < \ldots$ are integers and $(x_i)$ is the canonical basis of the block-subspace $q$. Then every block-sequence of $(x_{i_0}, x_{i_1}, \ldots)$ will belong to a set of the form $(\bigoplus_{i \in A_0} K_i) \times (\bigoplus_{i \in A_1} K_i) \times \cdots$.
Another interesting example

Lemma

A separable Banach space $E$ is $C$-isomorphic to $\ell_2$ if and only if every finite-dimensional subspace $F \subseteq E$ is $C$-isomorphic to $\ell_2^{\dim(F)}$. 
Lemma

A separable Banach space $E$ is $C$-isomorphic to $\ell_2$ if and only if every finite-dimensional subspace $F \subseteq E$ is $C$-isomorphic to $\ell_2^{\dim(F)}$.

Corollary

Let $E$ be a separable Banach space, non-isomorphic to $\ell_2$. Let $P$ be the set of (closed, infinite-dimensional) subspaces of $E$ that are not isomorphic to $\ell_2$, $\subseteq^*$ be the inclusion of subspaces up to finite dimension, and $d$ be the usual distance on $S_E$. Then $(P, S_E, d, \subseteq, \subseteq^*, \in)$ is an approximate Gowers space.
Corollary (dR. – Ferenczi)

Let $E$ be a separable Banach space, non-isomorphic to $\ell_2$, $\Delta$ be a sequence of positive numbers, $\varepsilon > 0$ and $\mathcal{X} \subseteq S_E$ be analytic. Then there exists a closed, infinite-dimensional subspace $X \subseteq E$, non-isomorphic to $\ell_2$, such that:

- either $X$ has an FDD $(F_n)_{n \in \mathbb{N}}$ with constant $\leq 1 + \varepsilon$, such that $d_{BM}(F_n, \ell_2^{\dim(F_n)}) \xrightarrow{n \to \infty} \infty$ and such that every normalized block-sequence of $(F_n)$ is in $\mathcal{X}^c$;
- or II has a winning strategy in $G_X((\mathcal{X})_\Delta)$, when I only plays subspaces that are non-isomorphic to $\ell_2$. 
Thank you for your attention!