Soft Square

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Section 1

The Objects of Study

Stationary Reflection

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Definition

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- If $S = \omega_2 \cap \mathrm{cof}(\omega)$ then S reflects at any ordinal of uncountable cofinality.
- If $S = \omega_2 \cap \mathrm{cof}(\omega_1)$ then S does not reflect. (Given $\alpha \in \omega_2 \cap \mathrm{cof}(\omega_1)$, consider a club $C \subset \alpha$ of order-type ω_1 and observe that $\lim C \cap S = \emptyset$.)

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- If κ is a singular cardinal of cofinality μ then *simultaneous reflection* holds for κ^+ if for every sequence $\langle S_i : i < \mu \rangle$ of stationary subsets of $\kappa^+ \cap \mathrm{cof}(\mu)$, there is some $\alpha < \kappa^+$ where the S_i's reflect simultaneously.

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Fact

If δ is supercompact and $\mathrm{cf}\,\kappa < \delta < \kappa^+$ then simultaneous stationary reflection holds for κ^+.

Square Sequences

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We say that $\square_{\kappa,\lambda}$ holds if there is a sequence $\langle \mathcal{C}_\alpha : \alpha \in \lim(\kappa^+) \rangle$ such that for all $\alpha \in \lim \kappa^+$:

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Proposition

Given a $\square_{\kappa,\lambda}$-sequence, there is no club $D \subset \kappa^+$ such that $\forall \alpha \in \lim D$, $D \cap \alpha \in \mathcal{C}_\alpha$.

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- $\square_\kappa^*$ is equivalent to existence of a special κ^+-Aronszajn tree.
- If $\square_\kappa^*$ holds then there is a second-countable non-metrizable topological space X such that $|X| = \kappa^+$ and every subspace of X of cardinality $< \kappa^+$ is metrizable.

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Theorem (Shelah)

If κ is a singular cardinal then there is a product of regular cardinals $\prod_{i < \mathrm{cf}\,\kappa} \kappa_i$ with $\sup_{i < \mathrm{cf}\,\kappa} \kappa_i = \kappa$ that carries a scale.

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Section 2

The Construction

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Fact

$\mathbb{S}(\kappa, < \lambda)$ *adds non-reflecting stationary sets in* $\kappa^+ \cap \mathrm{cof}(\mu)$ *for every* $\mu \leq \kappa$.

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Let G be $\mathbb{S}(\kappa, <\lambda)$-generic with $\bigcup G = \langle \mathcal{C}_\alpha : \alpha \in \lim \kappa^+ \rangle$, and let δ be an uncountable regular cardinal less than κ. In $V[G]$ we can define the poset $\mathbb{T}_\delta$ of closed bounded sets $c \subset \kappa^+$ of order-type less than δ such that $\forall \alpha \in \lim c$, $c \cap \alpha \in \mathcal{C}_\alpha$.

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Let $D(\mathbb{S}(\kappa, <\lambda) * \dot{\mathbb{T}}_\delta)$ be the set of pairs $(p, \check{c}) \in \mathbb{S}(\kappa, <\lambda) * \mathbb{T}_\delta$ where $c \in V$ and $\max \operatorname{dom} p = \max c$.

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$D(\mathbb{S}(\kappa, < \lambda) * \dot{\mathbb{T}}_\delta)$ is dense in $\mathbb{S}(\kappa, < \lambda) * \dot{\mathbb{T}}_\delta$ and is δ-directed closed.

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Fact

$\mathbb{T}_\delta$ destroys some of the stationary sets added by $\mathbb{S}(\kappa,<\lambda)$.

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Let G be $\mathbb{S}$-generic and let $\nu = (\kappa^+)^V$. If $f : \kappa^+ \to \mu$ is a partition in $V[G]$ for some $\mu < \kappa$ and $\tau < \delta$ are regular cardinals, then there is some $\xi < \mu$ such that $\Vdash_{\mathbb{T}_\delta}$ "$f^{-1}(\xi) \cap \mathrm{cof}(\tau)$ is stationary in ν".

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- If $(q, \emptyset) \leq (p^*, \emptyset)$ and $(q, \emptyset) \Vdash$ "$\dot{f}(\alpha^*) = \xi$", then this contradicts the previous point.

The Model with $\square_{\kappa,<\kappa}$ and no Very Good Scale [L.]

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- Very goodness for this scale fails at the point of reflection α.

...and κ can be $\aleph_\omega$

Theorem (L.)

Assuming the existence of a supercompact cardinal there is a model in which $\square_{\aleph_\omega,<\aleph_\omega}$ holds but there is no very good scale at $\aleph_\omega$.

$\square_{\kappa,<\kappa}$ and Simultaneous Reflection

Recall:

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- $\square_{\kappa,\lambda}$ for $\lambda < \kappa$ implies failure of simultaneous reflection at κ^+.

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Theorem (Cummings, Schimmerling)

If κ is a singular strong limit and $\square_{\kappa,<\kappa}$ holds, then there is a sequence $\langle S_i : i < \operatorname{cf}\kappa\rangle$ of stationary subsets of κ^+ and some $\mu < \kappa$ such that if the S_i's reflect simultaneously at α then $\operatorname{cf}\alpha > \mu$.

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Question

For singular κ, is $\square_{\kappa,<\kappa}$ consistent with simultaneous stationary reflection at κ^+?

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Theorem (L.)

Nope!

Section 3

Further Questions

Better Scales

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A scale $\langle f_\alpha : \alpha < \kappa^+ \rangle$ is a *better* scale if for every $\alpha < \kappa^+$ with $\operatorname{cf} \alpha > \operatorname{cf} \kappa$, there is a club $C \subset \alpha$ such that for every $\beta \in \lim C$, there is some $j < \operatorname{cf} \kappa$ such that for all $i \geq j$, $\gamma \in C \cap \beta$ implies $f_\beta(i) < f_\gamma(i)$.

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- *Very good scales are better scales.*
- $\square_\kappa^*$ *implies the existence of a better scale.*

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Fact (Hayut)

If κ is singular and $(\kappa^+)^{<\kappa^+} = \kappa^+$ then there is a $< \kappa^+$-strongly strategically closed poset that forces approachability at κ^+.

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Does $\mathbb{C}$ *add non-reflecting stationary subsets of* $\kappa^+ \cap \operatorname{cof}(\tau)$ *for* $\tau > \operatorname{cf} \kappa$*?*

Děkuji!