FORCING SQUARE SEQUENCES

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In the 1970's, Jensen proved that Gödel's constructible universe L satisfies a combinatorial principle called $\square_\kappa$ for every uncountable cardinal κ. Its significance is partially in that it clashes with the reflection properties of large cardinals—for example, if μ is supercompact and $\kappa \geq \mu$ then $\square_\kappa$ fails—and so it characterizes the minimality of L in an indirect way. Schimmerling devised an intermediate hierarchy of principles $\square_{\kappa,\lambda}$ for $\lambda \leq \kappa$ as a means of comparing a given model of set theory to L, the idea being that a smaller value of λ yields a model that is more similar to L at κ.

Cummings, Foreman, and Magidor proved that for any $\lambda < \kappa$, $\square_{\kappa,\lambda}$ implies the existence of a PCF-theoretic object called a very good scale for κ, but that $\square_{\kappa,\kappa}$ (usually denoted $\square^*_\kappa$) does not. They asked whether $\square_{\kappa,<\kappa}$ implies the existence of a very good scale for κ, and we resolve this question in the negative.

We will summarize the technical background of the problem, outline the construction of the model that serves as a counterexample, and discuss further avenues of research.