Non-measurability of the algebraic sums of sets of real numbers

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For sets $A, B \subseteq \mathbb{R}$, we define the algebraic sum

$$A + B = \{a + b \mid a \in A,\ b \in B\}.$$

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Theorem (Sierpiński 1920)

- *There exists sets $A, B \subseteq \mathbb{R}$ of measure zero, such that $A + B$ is non-measurabe.*
- *There exists sets $A, B \subseteq \mathbb{R}$ of the first category, such that $A + B$ doesn't have the Baire property.*

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Theorem (Rubel 1963)

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Theorem (Ciesieski-Fejzić-Freiling 2001, Kysiak 2005)

For every set $C \subseteq \mathbb{R}$ there exists a set $A \subseteq C$ such that $\lambda_*(A + A) = 0$ and $\lambda^*(A + A) = \lambda^*(C + C)$.

Notation

- $(X,+) = (\mathbb{R},+)$ or $(2^\omega, +_2)$
- $\mathcal{N}$ – σ-ideal of null subsets of X
- $\mathcal{M}$ – σ-ideal of meagre subsets of X

- $\exists^\infty_{n<\omega}\psi(n) \equiv$ "$\psi(n)$ holds for infinitely many $n<\omega$"
- $\forall^\infty_{n<\omega}\psi(n) \equiv$ "$\psi(n)$ holds for sufficiently large $n<\omega$"
- $\mathfrak{U}_{n<\omega}\psi(n) \equiv \{n<\omega \mid \psi(n)\} \in \mathcal{U}$,
  where $\mathcal{U}$ is a fixed non-principial ultrafilter on ω.

It's clear that $\forall^\infty_{n<\omega}\psi(n) \implies \mathfrak{U}_{n<\omega}\psi(n) \implies \exists^\infty_{n<\omega}\psi(n)$.

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On the question of Akbarov

In private communication with dr Marcin Kysiak, Sergei Akbarov asked

Question I

Assume $F \subseteq X$ is a meagre (null) subset of X. Does there exist a set $B \subseteq X$ such that $F + B$ doesn't have the Baire Property (is non-measurable)?

Question II

Assume $F \subseteq X$ is a meagre (null) subset of X. Does there exist an $x \in X$ and a dense subgroup $G \leq X$ such that $(F + x) \cap G = \emptyset$ and G doesn't have the Baire Property (is non-measurable)?

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From the affirmative answer to II, follows the affirmative answer to I:

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$F \in \mathcal{N}$, $G \cap (F + x) = \emptyset$, $G \leq X$ dense and non-measurable.

$$\Downarrow$$

$$(F + x) \cap (G - G) = \emptyset$$

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$$(F + x + G) \cap G = \emptyset$$

Both G and $F + x + G$ are non-measurable. □

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Partial answer

Under some additional assumptions, we can answer both affirmatively.

Theorem

- *If* $cov(\mathcal{M}) = cof(\mathcal{M})$, *then II - YES for the category.*
- *If* $cov(\mathcal{N}) = cof(\mathcal{N})$, *then II - YES for the measure.*

Question II for $\mathcal{N}$ is independent of ZFC

Theorem (Burke 1991 [4], Rosłanowski-Shelah 2016 [2])

It is relatively consistent with ZFC, that every meagre subgroup of X is null.

Corollary

II for measure – independent of ZFC

Proof.

Take $F \subseteq X$ dense G_δ of measure zero.
$(F+x)\cap G=\emptyset \implies G\in\mathcal{M} \implies G\in\mathcal{N}$. □

Theorem (Rosłanowski-Shelah 2016 [2])

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Theorem (Rosłanowski-Shelah 2016 [2])

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But not for $\mathcal{M}$

Theorem (K. 2017 [1])

For any meagre set $F \subseteq X$, there exists an $x \in X$, and a dense subgroup $G \leq X$ without the Baire property such that $(F + x) \cap G = \emptyset$.

Corollary

In particular, there exists a null, non-meagre subgroup – just take for F a meagre set of full measure.

Outline of the construction of $X = 2^\omega$

Ingredient number one:

Lemma (Bartoszyński [3])

Every meagre subset of 2^ω is contained in a meagre set of the form

$$F = \{x \in 2^\omega \mid \forall^\infty_{n<\omega}\, x \restriction I_n \neq v \restriction I_n\},$$

where $\{I_n\}_{n<\omega}$ is an interval partition of ω, and $v \in 2^\omega$.

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Ingredient number two:

Lemma (Rosłanowski-Shelah, 2016 [2])

Let $\{I_n\}_{n<\omega}$ be an interval partition. Then $G = \{x \in 2^\omega \,|\, \mathfrak{U}_{n<\omega}\, x \restriction I_n \equiv 0\}$ is a non-meagre dense subgroup of 2^ω.

Outline of the construction of $X = 2^\omega$

$$F = \{x \in 2^\omega \,|\, \forall^\infty_{n<\omega}\, x \restriction I_n \neq v \restriction I_n\},$$

$$G = \{x \in 2^\omega \,|\, \mathfrak{U}_{n<\omega}\, x \restriction I_n \equiv 0\}.$$

Therefore,

$$F + v \subseteq \{x \in 2^\omega \,|\, \forall^\infty_{n<\omega}\, x \restriction I_n \neq 0\},$$

and

$$G \subseteq \{x \in 2^\omega \,|\, \exists^\infty_{n<\omega}\, x \restriction I_n \equiv 0\}.$$

Clearly $(F + v) \cap G = \emptyset$. □

Summary

Question I

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	Question I	Question II
$\mathcal{I} = \mathcal{M}$	YES	YES
$\mathcal{I} = \mathcal{N}$	?	independent of ZFC

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What about the σ-ideal $\mathcal{E}$?

Definition

For $E \subseteq X$,
$E \in \mathcal{E}$ if and only if E can be covered by a countable family of compact null sets.

It can be shown that $\mathcal{E} \subsetneq \mathcal{N} \cap \mathcal{M}$.

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Theorem (Bartoszyński [3])

$E \in \mathcal{E}(2^\omega)$ if and only if

$$E \subseteq \{x \in 2^\omega \mid \forall^\infty_{n<\omega}\, x \upharpoonright I_n \in K_n\},$$

where $\{I_n\}_{n<\omega}$ is an interval partition of ω, $K_n \subseteq 2^{I_n}$ and $\forall_{n<\omega} \frac{|K_n|}{2^{|I_n|}} \leq 2^{-n}$.

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The following seems to be a reasonable question.

Problem (or challenge?)

Let $E \in \mathcal{E}(2^\omega)$. Does there necessarily exists a dense non-measurable subgroup $G \leq 2^\omega$, disjoint with some translation of E?

Thank You for attention!

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