Block Sequences with Projections into a Sequence of Happy Families

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The $k$-valued blocks $\text{Fin}_k$

**Definition**

Let $k \in \omega \setminus \{0\}$ unless stated otherwise.

1. For $p : \omega \to k + 1$ we let $\text{supp}(p) = \{n \in \omega : p(n) \neq 0\}$.

   \[ \text{Fin}_k = \{p : \omega \to k + 1 : \text{supp}(p) \text{ finite} \land k \in \text{range}(p)\}. \]
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3. For $a, b \in \text{Fin}_k$, we let $a < b$ denote $\text{supp}(a) < \text{supp}(b)$, i.e., $(\forall m \in \text{supp}(a))(\forall n \in \text{supp}(b))(m < n)$. A finite or infinite sequence $\langle a_i : i < m \leq \omega \rangle$ of elements of $\text{Fin}_k$ is in block-position if for any $i < j < m$, $a_i < a_j$. The set $(\text{Fin}_k)^\omega$ is the set of $\omega$-sequences in block-position, also called block sequences. For $n \geq 1$, the set $[\text{Fin}_k]^n_{<}$ is the set of $n$-sequences in block-position over $\text{Fin}_k$. 
Two operations on $\text{Fin}_j$

Definition

(4) For $k \geq 1$, $a, b \in \text{Fin}_k$, we define the partial semigroup operation $+$ as follows: If $\text{supp}(a) < \text{supp}(b)$, then $a + b \in \text{Fin}_k$ is defined. We let $(a + b)(n) = a(n) + b(n)$. Otherwise $a + b$ is undefined. Thus

$$a + b = a \upharpoonright \text{supp}(a) \cup b \upharpoonright \text{supp}(b) \cup 0 \upharpoonright (\omega \setminus (\text{supp}(a) \cup \text{supp}(b)))$$.
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(5) For any $k \geq 2$ we define on $\text{Fin}_k$ the Tetris operation: $T: \text{Fin}_k \to \text{Fin}_{k-1}$ by $T(p)(n) = \max\{p(n) - 1, 0\}$. 
Definition

(6) Let \( B \subseteq \text{Fin}_k \) be min-unbounded. We let

\[
\text{TFU}_k(B) = \left\{ T^{(j_0)}(b_{n_0}) + \cdots + T^{(j_\ell)}(b_{n_\ell}) : \right.
\]

\[
\ell \in \omega \setminus \{0\}, \ b_{n_i} \in B, \ b_{n_0} < \cdots < b_{n_\ell},
\]

\[
(\exists r \leq \ell j_r = 0)
\]

be the partial subsemigroup of \( \text{Fin}_k \) generated by \( B \). We call \( B \) a TFU\(_k\)-set if \( B = \text{TFU}_k(B) \).
The condensation order

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(8) We define the \textit{past-operation}: Let \( \bar{a} \in (\text{Fin}_k)^\omega \) and \( p \in \text{Fin}_k \).

\[
(\bar{a} \text{ past } p) = \langle a_i : i \geq i_0 \rangle
\]

with \( i_0 = \min \{ i : \text{supp}(a_i) > p \} \).
Lemma

If there is \( \bar{c} \sqsubseteq_k \bar{a}, \bar{b} \), then there is a largest one and it can be computed by finite initial segments.

Proof.

We define a well-order (of type \( \omega \)) \( \leq_{\text{lex}, \text{Fin}_k} \) on the set \( \text{Fin}_k \) via a well-order (of type \( \omega \)) \( \leq_{\text{lex}, \text{Fin}_k} \) on the set \( \text{Fin}_k \) via

\[
a <_{\text{lex}, \text{Fin}_k} b \text{ if } \max(\text{supp}(a)) < \max(\text{supp}(b)) \text{ or } \\
(\max(\text{supp}(a)) = \max(\text{supp}(b)) \text{ and there is an } m \text{ such that } \text{a} \upharpoonright m = \text{b} \upharpoonright m \text{ and } a(m) > b(m). \]

For a non-empty set \( X \subseteq \text{Fin}_k \) we let \( \min_{\text{Fin}_k}(X) \) be the \( \leq_{\text{lex}, \text{Fin}_k} \)-least element of \( X \). We let

\[
c_0 = \min_{\text{lex}, \text{Fin}_k} (\text{TFU}_k(\bar{a}) \cap \text{TFU}_k(\bar{b})), \\
c_{n+1} = \min_{\text{lex}, \text{Fin}_k} (\text{TFU}_k(\bar{a} \text{ past } c_n) \cap \text{TFU}_k(\bar{b} \text{ past } c_n))
\]
A subspace of $(\text{Fin}_k)^\omega$—Fixing $PP$

Definition

We fix parameters as follows. Let $k \geq 1$. Fix $P_{\min}, P_{\max} \subseteq \{1, \ldots, k\}$. Let $PP = \{(i, x) : x \in \{\text{min, max}\}, i \in P_x\}$ and let

$$\bar{\mathcal{R}} = \{(\iota, \mathcal{R}_\iota) : \iota \in PP\}$$

be a $PP$-sequence of pairwise nnc Ramsey ultrafilters (pairwise nnc selective coideals, i.e. happy families, would suffice for the pure decision property and properness). We also name the end segments for $1 \leq j \leq k$:

$$\bar{\mathcal{R}} \upharpoonright \{j, \ldots, k\} = \{(\iota, \mathcal{R}_\iota) : \iota = (i, x) \in PP \land i \in \{j, \ldots, k\}\}.$$
A subspace of \((\text{Fin}_k)^\omega\)

**Definition**

We let \((\text{Fin}_k)^\omega(\bar{R})\) denote the set of \(\text{Fin}_k\)-blocksequences \(\bar{a}\) with the following properties:

\[
\begin{align*}
(\forall i \in P_{\min})\{\min(a^{-1}_n[{i}]) : n \in \omega\} & \in R_{i,\min} \land \\
(\forall i \in P_{\max})\{\max(a^{-1}_n[{i}]) : n \in \omega\} & \in R_{i,\max} \land \\
(\forall s \in \text{TFU}_k(\bar{a}))\left(\min(s^{-1}[\{1\}]) < \min(s^{-1}[\{2\}]) < \cdots < \min(s^{-1}[\{k\}])
\right. \\
& \left. \max(s^{-1}[\{k\}]) < \max(s^{-1}[\{k - 1\}]) < \cdots < \max(s^{-1}[\{1\}]). \right)
\]

\(0.1\)

If \((i, x) \in \{1, \ldots, k\} \times \{\min, \max\} \setminus PP\), we leave the term \(x(s^{-1}[\{i\}])\) out of the equation \((0.1)\).
Lemma

There are $\sqsubseteq^*_k$-incompatible elements in $(\text{Fin}_k)^\omega(\bar{\mathcal{R}})$. Indeed, there are $\bar{a}, \bar{b} \in (\text{Fin}_k)^\omega(\bar{\mathcal{R}})$ such that for any $j = 0, \ldots, k - 1$ the $\text{Fin}_{k-j}$-block-sequences $T^{(j)}[\bar{a}]$ and $T^{(j)}[\bar{b}]$ are $\sqsubseteq^*_{k-j}$-incompatible.
A common strengthening of a theorem by Gowers and a theorem by Blass

The special case of $PP = \{(1, \min), (1, \max)\}$ was proved by Blass in 1987, the case $PP = \emptyset$ and arbitrary finite $k$ by Gowers in 1992.

**Theorem**

Let $k$, $PP$, $\bar{\mathcal{R}}$ be as above. Let $\bar{a} \in (\text{Fin}_k)\omega(\bar{\mathcal{R}})$ and let $c$ be a colouring of $\text{TFU}_k(\bar{a})$ into finitely many colours. Then there is a $\bar{b} \subseteq_k \bar{a}$, $\bar{b} \in (\text{Fin}_k)\omega(\bar{\mathcal{R}})$ such that $\text{TFU}_k(\bar{b})$ is $c$-monochromatic.
**Sketch: Proof via Galvin-Glazer technique**

**Definition**

Given $k, P_{\text{min}}, P_{\text{max}}$ and $\bar{R}$ as above, we define

$$
\gamma(\text{Fin}_k(\bar{R})) = \{ \mathcal{U} : \mathcal{U} \text{ is a min-unbounded ultrafilter over } \text{Fin}_k \}
$$

$$
(\forall i \in P_{\text{min}})(\hat{\min}_i(\mathcal{U}) = R_{i,\text{min}}) \land
(\forall i \in P_{\text{max}})(\hat{\min}_i(\mathcal{U}) = R_{i,\text{max}}),
$$

endowed with the topology given by the basic open sets

$$
\left\{ \left\{ \mathcal{U} \in \gamma(\text{Fin}_k(\bar{R})) : A \in \mathcal{U} \right\} : A \subset \text{Fin}_k, \right. 
x(s^{-1}[\{i\}]) : s \in A \right\} \in R_{i,x}. 
$$

The space $\gamma(\text{Fin}_k(\bar{R}))$ a compact Hausdorff space.
For work with semigroups of ultrafilters we temporarily have to choose $PP$ in a narrower sense. The reason is the claim part of Def. and Lemma below. We do not know how to handle missing $i$ in the sequence of $\mathcal{R}_{i,\min}$’s or in the sequence of $\mathcal{R}_{i,\max}$’s in the claim.

**Definition**

For any $k \geq 1$, a reservoir of indices $PP$ of the strict form is one of the following three types: $PP = \{(i, \min), (i, \max) : 1 \leq i \leq k\}$, $PP = \{(i, \min) : 1 \leq i \leq k\}$, $PP = \{(i, \min) : 1 \leq i \leq k\}$. 

Strict $PP$’s
A lift of the tetris operation

Definition and Lemma

Again we work with strict PP. For $2 \leq j \leq k$, we write

$$T[X] = \{T(a) : a \in X\} \text{ for } X \subseteq \text{Fin}_j(\bar{R} \restriction \{k - j + 1, k\})$$ \text{ and } $$T[\bar{a}] = \langle T(a_n) : n \in \omega \rangle \text{ for } \bar{a} \in (\text{Fin}_j)^\omega(\bar{R} \restriction \{k - j + 1, k\}).$$

The lift of the tetris operation

$$\dot{T} : \gamma(\text{Fin}_j(\bar{R} \restriction \{k-j+1, \ldots, k\})) \to \gamma(\text{Fin}_{j-1}(\bar{R} \restriction \{k-j+2, \ldots, k\}))$$

is defined via

$$\dot{T}(\mathcal{U}) = \{T[X] : X \in \mathcal{U}\}.$$
A lift of the partial semigroup operation +

Definition and Lemma

Let $k$, $PP$ and $\tilde{R}$ be as above, with strict $PP$. We define $\dot{+}$ on $(\bigcup_{j=1}^{k} \gamma(\text{Fin}_j)(\tilde{R} \upharpoonright \{k - j + 1, \ldots, k\}))^2$ as follows.

\[
\dot{+} : \gamma(\text{Fin}_i(\tilde{R} \upharpoonright \{k - i + 1, \ldots, k\})) \times \gamma(\text{Fin}_j(\tilde{R} \upharpoonright \{k - j + 1, \ldots, k\})) \\
\rightarrow \gamma\text{Fin}_{\max\{i,j\}}(\tilde{R} \upharpoonright \{k - \max(i,j) + 1, \ldots, k\})
\]

is defined as

\[
\mathcal{U} + \mathcal{V} = \left\{ X \subseteq \text{Fin}_{\max\{i,j\}}(\tilde{R} \upharpoonright \{k - \max(i,j) + 1, \ldots, k\}) \right. \\
: \left. \{ s : \{ t : s + t \in X \} \in \mathcal{V} \right\} \in \mathcal{U} \right\}.
\]
Lemma

let $k$, $PP$, $\bar{R}$ be as above, not necessarily strict. Here the strict form of $PP$ is not needed. Any $\sqsubseteq_k$-descending sequence $\langle \bar{c}_n : n \in \omega \rangle$ in $(\text{Fin}_k)^\omega(\bar{R})$ has a diagonal lower bound $\bar{b} \in (\text{Fin}_k)^\omega(\bar{R})$

$$(\forall n \in \omega)((\bar{b} \text{ past } b_n) \sqsubseteq_k \bar{c}_{\max(\text{supp}(b_n))+1}).$$

such that each $b_{n+1}$ is an element of $\{c_{\ell_{n+1},m} : m \in \omega\}$ for some $\ell_{n+1} > \max(\text{supp}(b_n))$ and $b_0$ is an element of $\{c_{\ell_0,m} : m \in \omega\}$ for some $\ell_0$. 
A \( k \)-sequence of very good idempotent ultrafilters

Lemma

(Lemma 2.24, Todorcevic, Ramsey Spaces) Let \( k, PP, \bar{R} \) be as above, with full \( PP \). For any \( k \geq j \geq 1 \), and \( \bar{a} \in (\text{Fin}_k)^\omega(\bar{R}) \) there is an idempotent \( U_j \in \gamma(\text{Fin}_j(\bar{R} \upharpoonright \{k+j-1, \ldots, k\})) \) such that for all \( 1 \leq i \leq j \leq k \)

1. \( U_j \cdot U_i = U_j \),
2. \( \bar{T}^{(j-i)}(U_j) = U_i \).
3. \( \bar{T}^{(i-1)}(\bar{a}) \in U_{k-i+1} \).
A useful notion of forcing

Definition
We let \( k, \, PP, \, \bar{R} \) be as above, not necessarily strict. In the Gowers–Matet forcing with \( \bar{R}, \, M_k(\bar{R}) \), the conditions are pairs \((s, \bar{c})\) such that \( s \in \text{Fin}_k \) and \( \bar{c} \in (\text{Fin}_k)^\omega(\bar{R}) \) and \( \text{supp}(s) < \text{supp}(c_0) \).

The forcing order is: \((t, \bar{b}) \leq (s, \bar{a})\) if \( t = s + s' \) and \( s' \in \text{TFU}_k(\bar{a}) \) and \( \bar{b} \sqsubseteq_k (\bar{a} \, \text{past} \, s') \).

Definition
For \((s, \bar{a}), (t, \bar{b}) \in M_k(\bar{R})\) and \( n \in \omega \) we let \((s, \bar{a}) \leq_n (t, \bar{b})\) if \( s = t \) and \( a_i = b_i \) for \( i < n \).

Lemma
\( M_k(\bar{R}) \) has the pure decision property, i.e., for any \( \varphi \in \mathcal{L}(\varepsilon) \), \((s, \bar{a}) \in M_k(\bar{R}) \) \exists(s, \bar{b}) \leq (s, \bar{a}) ((s, \bar{b}) \vDash \varphi \lor (s, \bar{b}) \vDash \neg \varphi). \)
Stepping up to finite dimensions

Since the space \((\text{Fin}_k)^\omega(\bar{R})\) is stable, we can step up the Milliken–Taylor style to higher finite arities:

**Theorem**

Let \(n \in \omega \setminus \{0\}\) and \(\bar{a} \in (\text{Fin}_k)^\omega(\bar{R})\) and let \(c\) be a colouring of \([\text{TFU}_k(\bar{a})]^n_\prec\) into finitely many colours. Then there is a \(\bar{b} \subseteq_k \bar{a}\), \(\bar{b} \in (\text{Fin}_k)^\omega(\bar{R})\) such that \([\text{TFU}_k(\bar{b})]^n_\prec\) is \(c\)-monochromatic.