The Mardešić conjecture and free products of Boolean algebras

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The conjecture

If $L_1, L_2, \ldots, L_d$ are compact lines and there is a continuous

$$L_1 \times L_2 \times \ldots \times L_d \xrightarrow{\text{onto}} K_1 \times K_2 \times \ldots \times K_d \times K_{d+1},$$

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Theorem

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$$\text{free}-\text{dim}\big(K_1 \times K_2 \times \ldots \times K_d \times K_{d+1}\big) \geq d + 1.$$

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$$|\mathscr{C}| \leq 2\big(|\mathscr{C}_1| + |\mathscr{C}_2|\big).$$

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4. free-dim$(K) = 1$ for every infinite metric compactum.

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Theorem

If $\mathfrak{A}_1, \ldots, \mathfrak{A}_d$ are uncountable Boolean algebras and $\mathfrak{A}_{d+1}$ is infinite then the free product

$$\mathfrak{A}_1 \otimes \ldots \otimes \mathfrak{A}_d \otimes \mathfrak{A}_{d+1} \not\hookrightarrow \mathfrak{B}_1 \otimes \ldots \otimes \mathfrak{B}_d$$

for any interval algebras $\mathfrak{B}_i$.

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Proof.

For any finite $F \subseteq \Gamma$, let $\mathscr{C}_F$ be a finite closed cover of $K_{\mathfrak{B}}$ determined by the atoms of $\langle F \rangle$.

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Check, using the Sauer-Shelah lemma, that the family

$$\mathbb{C} = \left\{ \mathscr{C}_F : F \in [\Gamma]^{<\omega} \right\},$$

witnesses that $\text{free-dim}(K_{\mathfrak{B}}) \leq d$.

The Sauer-Shelah lemma

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Lemma

Let N, d be natural numbers with $0 \leq d < N$ and let $T = \{1, 2, \ldots, N\}$. Then for every family $C \subseteq 2^T$ with

$$|C| > \binom{N}{0} + \binom{N}{1} + \cdots + \binom{N}{d},$$

there exists a set $S \subseteq T$ with $|S| = d + 1$ such that $\{f|_S : f \in C\} = 2^S$.