Edinburgh topology on 2^κ

Wolfgang Wohofsky

joint work with Peter Holy, Marlene Koelbing, and Philipp Schlicht

Universität Hamburg

wolfgang.wohofsky@gmx.at

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Let κ be an uncountable regular cardinal.

- κ successor cardinal
- κ inaccessible cardinal

We consider (the generalized Cantor space) 2^κ, equipped with two different topologies.

Usually, 2^κ is equipped with the following topology:

Definition (Bounded topology)

...is generated by $\{[s] : s \in 2^{<\kappa}\}$

...where $[s] = \{x \in 2^\kappa : s \subseteq x\}$

Definition (Bounded topology)

...is generated by the basic clopen sets $\{[s] : s \in 2^{<\kappa}\}$

...where $[s] = \{x \in 2^\kappa : s \subseteq x\}$

- $X \subseteq 2^\kappa$ is open if $X = \bigcup_{i \in I} [s_i]$
  - X is closed if $2^\kappa \setminus X$ is open (iff $X = [T]$ for some tree $T \subseteq 2^{<\kappa}$)
- X is dense if for each $s \in 2^{<\kappa}$ $(X \cap [s] \neq \emptyset)$
  - If X is open dense then for each s there is $t \supseteq s$ with $[t] \subseteq X$
- X is nowhere dense if for each s there is $t \supseteq s$ with $X \cap [t] = \emptyset$
- X is meager if $X \subseteq \bigcup_{i<\kappa} A_i$ with each A_i (closed) nowhere dense
- Baire Category: intersection of κ many open dense sets is dense
  - 2^κ is not meager
  - $[s]$ is not meager

Definition (Bounded topology)

…is generated by $\{[s] : s \in 2^{<\kappa}\}$

…where $[s] = \{x \in 2^\kappa : s \subseteq x\}$

s is defined on a "small" domain

"small" means bounded, i.e., of size less than κ

What about other ideals than the ideal of bounded sets?

Let $\heartsuit$ denote the set of
partial functions f from κ to 2 with $\mathrm{dom}(f)$ non-stationary.

Definition (Edinburgh topology (or non-stationary topology))

…is generated by $\{[f] : f \in \heartsuit\}$

…where $[f] = \{x \in 2^\kappa : f \subseteq x\}$

Let $\heartsuit$ denote the set of partial functions f from κ to 2 with $\mathrm{dom}(f)$ non-stationary.

Definition (Edinburgh topology (or non-stationary topology))

…is generated by $\{[f] : f \in \heartsuit\}$

…where $[f] = \{x \in 2^\kappa : f \subseteq x\}$

- $X \subseteq 2^\kappa$ is Edinburgh open if $X = \bigcup_{i \in I} [f_i]$ (with $f_i \in \heartsuit$)
- Every open set is also open in the Edinburgh topology (i.e., the Edinburgh topology refines the bounded topology).
- Each "Edinburgh cone" $[f]$ (i.e., Edinburgh basic clopen set) is closed (but usually not open)
- Edinburgh nowhere dense
- Edinburgh meager
- Baire category: intersection of κ many Edinburgh open dense sets is Edinburgh dense

Are the Borel sets and the Edinburgh Borel sets the same?

Proposition

There are exactly 2^{2^κ} many Edinburgh open sets.

So the answer to the above question is no: there are even more Edinburgh open sets than (usual) Borel sets.

Basic properties of Edinburgh nowhere dense sets:

- every set of size $< 2^\kappa$ is Edinburgh nowhere dense
- there is an Edinburgh nowhere dense set of size 2^κ
  - there is even a closed such set (actually of the form $[T]$ with T a perfect subtree of $2^{<\kappa}$ with uniform splitting):

$$\{x \in 2^\kappa : x(\alpha) = x(\alpha + 1) \text{ for each even } \alpha < \kappa\}$$

Edinburgh nowhere dense $\subsetneq$ Edinburgh meager ?

Theorem

Assume κ is inaccessible or $\diamondsuit_\kappa$ holds.
Then every Edinburgh meager set is Edinburgh nowhere dense.

Proposition

If $f \in \heartsuit$ and $|\mathrm{dom}(f)| = \kappa$, then $[f]$ is closed nowhere dense.
So there is a meager set which is not Edinburgh meager.

What about the other direction ???

Lemma

Assume κ is inaccessible or $\diamondsuit_\kappa$ holds.
Then each co-meager set contains an Edinburgh cone $[f]$:

$$\text{given } (D_\alpha)_{\alpha<\kappa} \text{ open dense} \quad \exists f \in \heartsuit \quad \bigcap_{\alpha<\kappa} D_\alpha \supseteq [f]$$

Theorem

Assume X has the Baire property and the conclusion of the lemma holds.
Then "X Edinburgh meager" implies "X meager".

For $a, y \in [\kappa]^\kappa$, we say that a splits y if $a \cap y$ and $(\kappa \setminus a) \cap y$ are of size κ.

Definition

A reaping family on κ is a set $\mathcal{R} \subseteq [\kappa]^\kappa$ such that

no $a \in [\kappa]^\kappa$ splits all $y \in \mathcal{R}$.

$\mathfrak{r}(\kappa)$ is the smallest size of a reaping family on κ.

Theorem (κ inaccessible)

Assume $\mathfrak{r}(\kappa) = 2^\kappa$.
Then there is an Edinburgh nowhere dense set which is not meager.

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Assume $\mathfrak{r}(\kappa) = 2^\kappa$.
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Let $\mathfrak{S}$ denote the set of partial functions from κ to 2 with $|\mathrm{dom}(f)| = \kappa$.

Definition

$\mathfrak{ph}(\kappa)$ is the smallest size of a family $\mathcal{F} \subseteq \mathfrak{S}$ such that $\bigcup_{f \in \mathcal{F}} [f] = 2^\kappa$.

Lemma (κ inaccessible)

$\mathfrak{ph}(\kappa)$ is the smallest size of a family $\mathcal{F} \subseteq \mathfrak{S}$ such that $\bigcup_{f \in \mathcal{F}} [f]$ is co-meager.

Lemma ($|2^{<\kappa}| = \kappa$)

$\mathfrak{ph}(\kappa) = \mathfrak{r}(\kappa)$.

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