Upper Namioka property of multi-valued mappings

Chernivtsi - Kielce

Namioka and co-Namioka spaces

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Theorem 1 (I.Namioka, 1974)

Let X be a strongly countably complete space, Y be a compact space and $f : X \times Y \to \mathbb{R}$ be a separately continuous function. Then there exists an everywhere dense G_δ-set A in X such that the function f is continuous at every point of the set $A \times Y$.

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- any Valdivia compact is co-Namioka (A. Bouziad, 1994);
- any linearly ordered compact is co-Namioka (M., 2007);
- class of compact co-Namioka spaces is closed over products (A. Bouziad, 1996).

Upper Namioka and co-Namioka spaces

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Let X be a Baire s., Y be a second countable s. and $F \in LU(X, Y)$ be a compact-valued mapping. Then there $\exists$ a dense in X G_δ-set $A \subseteq X$ such that F is jointly upper semi-continuous at each point of $A \times Y$.

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Multi-valued map $F : X \times Y \to \mathbb{R}$ has the upper Namioka property if $\exists$ dense in X G_δ-set $A \subseteq X$ such that F is jointly upper semi-continuous at every point of the set $A \times Y$.

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Upper Namioka spaces are trivial

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Proposition 1.

Let X and Y be topological spaces, $(x_s : s \in S)$ be a family of non-isolated distinct points $x_s \in X$ such that every set $\{x_s\}$ is closed in X and $(G_s : s \in S)$ be a family of nonempty functionally open pairwise disjoint sets $G_s \subseteq Y$. Then there exists a compact-valued mapping $F : X \times Y \to [0, 1]$ which is lower semi-continuous with respect to the first variable, continuous with respect to the second one and for every $s \in S$ there exists $y_s \in G_s$ such that F_{y_s} is upper discontinuous at x_s.

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Corollary 1.

Let X be a T_1-space. Then the following conditions are equivalent:
(i) X is upper Namioka space;
(ii) the set A of all isolated points of X is dense in X.

Upper co-Namioka and co-Namioka spaces are completely different

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Proposition 2.

Let $|S| = \aleph_1$, X – s. of all $x : S \to \{0,1\}$ with at most countable support, with the topology of the uniform convergence on countable sets and Y – t. s. with a strictly increasing sequence $(H_\xi : 0 \leq \xi \leq \omega_1)$ of closed in Y sets H_ξ. Then $\exists$ a mapping $F \in LU(X, Y)$ such that $\forall$ $x \in X$ $\exists$ $y_x \in Y$ such that F_{y_x} is upper discontinuous at x.

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Corollary 2.

(i) Every subset of upper co-Namioka space is separable.

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(i) Every subset of upper co-Namioka space is separable.
(ii) Every well-ordered upper co-Namioka compact space is metrizable.

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(iii) Every upper co-Namioka Valdivia compact space is metrizable.

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Corollary 2.

(i) Every subset of upper co-Namioka space is separable.
(ii) Every well-ordered upper co-Namioka compact space is metrizable.
(iii) Every upper co-Namioka Valdivia compact space is metrizable.
(iv) There exists a family $(Y_s : s \in S)$ of upper co-Namioka spaces Y_s such that the product $Y = \prod\limits_{s\in S} Y_s$ is not upper co-Namioka.

Linearly ordered upper co-Namioka spaces

Corollary 3.

Let Y be a linearly ordered compact such that Y^2 is upper co-Namioka. Then Y is metrizable.

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Question 3.

Does there exist a non-metrizable linearly ordered upper co-Namioka space?

Definition 4.

Let $(X, <)$ be a linearly ordered set and $A \subseteq X$. We consider on the set

$$X_A = (X \setminus A) \bigcup \left(\bigcup_{x \in A} \{(x, 0), (x, 1)\} \right)$$

the following order:

$$u \prec v$$

– if $u, v \in X \setminus A$ and $u < v$;
– if $u \in X \setminus A$, $v \in \{(a, 0), (a, 1)\}$ for some $a \in A$ and $u < a$;
– if $u \in \{(a, 0), (a, 1)$ for some $a \in A$, $v \in X \setminus A$ and $a < v$;
– if $u \in \{(a, 0), (a, 1)$ and $v \in \{(a, 0), (a, 1)\}$ for some $a, b \in A$ with $a < b$;
– if $x = (a, 0)$ and $y = (a, 1)$ for some $a \in A$.
The linearly ordered set $(X_A, \prec)$ we shall call by doubling of X through A.

Notice that $[0,1]_{[0,1]}$ is the "two arrows space".

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Every separable linearly ordered space Y is homeomorphic to the doubling of some space $X \subseteq [0, 1]$ through a set $A \subseteq X$.

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Proposition 4.

Let $Z \subseteq [0,1]$ be a compact and $A \subseteq Z$ be such that the space $Y = Z_A$ is upper co-Namioka. Then A is always of the first category.

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The double arrow space is not upper co-Namioka space.

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Proposition 5.

There exist a Namioka space X, a co-Namioka space Y and a compact-valued mapping $F \in LU(X, Y)$ such that F has not the upper Namioka property.

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Question 5.

Let $Z \subseteq [0,1]$ be a compact and $A \subseteq Z$ be an always of the first category. Is it true that the space $X = Z_A$ is upper co-Namioka?

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Let $Z \subseteq [0,1]$ be a compact and $A \subseteq Z$ be an always of the first category. Is it true that the space $X = Z_A$ is upper co-Namioka?

Question 6.

Is it true that the product of finite (countable) family of upper co-Namioka spaces is upper co-Namioka?

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Thank you