A Parallel Metrization Theorem

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(Lviv and Kielce)

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Parallel sets in metric spaces

In this talk I shall present a solution of one question asked on Mathoverflow by `user116515`.

The question concerns parallel sets in metric spaces.

Definition

Two non-empty sets A, B in a metric space (X, d) are called *parallel* if

$$d(a, B) = d(A, B) = d(A, b) \quad \text{for any } a \in A \text{ and } b \in B.$$

Here $d(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\}$
and $d(x, B) = d(B, x) := d(\{x\}, B)$ for $x \in X$.

Observe that two closed parallel sets A, B is a metric space are either disjoint or coincide.

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A MO problem on parallel metrics

Definition

Let $\mathcal{C}$ be a family of closed subsets of a topological space X. A metric d on X is called *$\mathcal{C}$-parallel* if any two sets $A, B \in \mathcal{C}$ are parallel with respect to the metric d.

A family $\mathcal{C}$ of subsets of X is called a *compact cover* of X if $X = \bigcup \mathcal{C}$ and each set $C \in \mathcal{C}$ is compact.

Problem (MO)

For which compact covers $\mathcal{C}$ of a topological space X the topology of X is generated by a $\mathcal{C}$-parallel metric?

Example

The Euclidean metric on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| \le 1\}$ is parallel with respect to the cover $\mathcal{C} = \{C_r : r \in [0,1]\}$ of $\mathbb{D}$ by circles $C_r = \{z \in \mathbb{C} : |z| = r\}$.

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Continuity of families

A metric generating the topology of a given topological space is called *admissible*.

Let $\mathcal{C}$ be a cover $\mathcal{C}$ of a set X. A subset $A \subset X$ is called *$\mathcal{C}$-saturated* if A coincides with its *$\mathcal{C}$-saturation*

$$[A]_{\mathcal{C}} := \bigcup \{C \in \mathcal{C} : A \cap C \neq \emptyset\}.$$

A family $\mathcal{C}$ of subsets of a topological space X is called

- *lower semicontinuous* if for any open set $U \subset X$ its $\mathcal{C}$-saturation $[U]_{\mathcal{C}}$ is open in X;
- *upper semicontinuous* if for any closed set $F \subset X$ its $\mathcal{C}$-saturation $[F]_{\mathcal{C}}$ is closed in X;
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Main Theorem

For a compact cover $\mathcal{C}$ of a metrizable topological space X the following conditions are equivalent:

1. *the topology of X is generated by a $\mathcal{C}$-parallel metric;*
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Proof. (1) $\Rightarrow$ (2) Assume that d is an admissible $\mathcal{C}$-parallel metric on X. The disjointness of the cover $\mathcal{C}$ follows from the obvious observation that two closed parallel sets in a metric space are either disjoint or coincide.

Proof of the parallel metrization theorem

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Proof of Main Theorem (1) $\Rightarrow$ (2)

To see that $\mathcal{C}$ is lower semicontinuous, fix any open set $U \subset X$ and consider its $\mathcal{C}$-saturation $[U]_{\mathcal{C}}$. To see that $[U]_{\mathcal{C}}$ is open, take any point $s \in [U]_{\mathcal{C}}$ and find a set $C \in \mathcal{C}$ such that $s \in C$ and $C \cap U \neq \emptyset$. Fix a point $u \in U \cap C$ and find $\varepsilon > 0$ such that the ε-ball $B(u, \varepsilon) = \{x \in X : d(x, u) < \varepsilon\}$ is contained in U. We claim that $B(s, \varepsilon) \subset [U]_{\mathcal{C}}$. Indeed, for any $x \in B(s, \varepsilon)$ we can find a set $C_x \in \mathcal{C}$ containing x and conclude that $d(C_x, u) = d(C_x, C) \leq d(x, s) < \varepsilon$ and hence $C_x \cap U \neq \emptyset$ and $x \in C_x \subset [U]_{\mathcal{C}}$.

To see that $\mathcal{F}$ is lower semicontinuous, fix any closed set $F \subset X$ and consider its $\mathcal{C}$-saturation $[F]_{\mathcal{C}}$. To see that $[F]_{\mathcal{C}}$ is closed, take any point $s \in X \setminus [F]_{\mathcal{C}}$ and find a set $C \in \mathcal{C}$ such that $s \in C$. It follows from $s \notin [F]_{\mathcal{C}}$ that $C \cap F = \emptyset$ and hence $\varepsilon := d(C, F) > 0$ by the compactness of C. We claim that $B(s, \varepsilon) \cap [F]_{\mathcal{C}} = \emptyset$. Assuming the opposite, we can find a point $x \in B(s, \varepsilon) \cap [F]_{\mathcal{C}}$ and a set $C_x \in \mathcal{C}$ such that $x \in C_x$ and $C_x \cap F \neq \emptyset$. Fix a point $z \in C_x \cap F$ and observe that

$$d(C, F) \leq d(C, z) = d(C, C_x) \leq d(s, x) < \varepsilon = d(C, F),$$

which is a desired contradiction.

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Proof of Theorem (2) $\Rightarrow$ (1)

The proof of the implication (2) $\Rightarrow$ (1) is more difficult.

Assume that $\mathcal{C}$ is disjoint and continuous.

Fix any admissible metric $\rho \le 1$ on X.

Let $\mathcal{U}_0(C) = \{X\}$ for every $C \in \mathcal{C}$.

Claim

For every $n \in \mathbb{N}$ and every $C \in \mathcal{C}$ there exists a finite cover $\mathcal{U}_n(C)$ of C by open subsets of X such that

(i) *each set $U \in \mathcal{U}_n(C)$ has ρ-diameter $\le \frac{1}{2^n}$;*

(ii) *if a set $A \in \mathcal{C}$ meets some set $U \in \mathcal{U}_n(C)$, then $A \subset \bigcup \mathcal{U}_n(C)$ and A meets each set $U' \in \mathcal{U}_n(C)$.*

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Proof of the Claim

The paracompactness of the metrizable space X yields a locally finite cover $\mathcal{V}$ of X by open sets of ρ-diameter $< \frac{1}{2^n}$.

For every compact set $C \in \mathcal{C}$ consider the finite subfamily $\mathcal{V}(C) := \{V \in \mathcal{V} : V \cap C \neq \emptyset\}$ of the locally finite cover $\mathcal{V}$.

Since the cover $\mathcal{C}$ is upper semicontinuous, the $\mathcal{C}$-saturated set $F_C = [X \setminus \bigcup \mathcal{V}(C)]_{\mathcal{C}}$ is closed and disjoint with the set C.

Since $\mathcal{C}$ is lower semi-continuous, for any open set $V \in \mathcal{V}(C)$ the set $[V]_{\mathcal{C}}$ is open and hence $W[C] := \bigcap_{V \in \mathcal{V}(C)} [V]_{\mathcal{C}} \setminus F_C$ is an open $\mathcal{C}$-saturated neighborhood of C in X.

Put $\mathcal{U}_n(C) := \{W(C) \cap V : V \in \mathcal{V}(C)\}$ and observe that $\mathcal{U}_n$ satisfies the condition

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Put $\mathcal{U}_n(C) := \{W(C) \cap V : V \in \mathcal{V}(C)\}$ and observe that $\mathcal{U}_n$ satisfies the condition

(i) each set $U \in \mathcal{U}_n(C)$ has ρ-diameter $\leq \frac{1}{2^n}$.

Proof of the Claim (continuation)

Let us show that the cover $\mathcal{U}_n(C)$ satisfies the condition

(ii) if a set $A \in \mathcal{C}$ meets some set $U \in \mathcal{U}_n(C)$, then $A \subset \bigcup \mathcal{U}_n(C)$ and A meets each set $U' \in \mathcal{U}_n(C)$.

Assume that a set $A \in \mathcal{C}$ meets some set $U \in \mathcal{U}_n(C)$.

First we show that $A \subset \bigcup \mathcal{U}_n(C)$.

Find a set $V \in \mathcal{V}(C)$ such that $U = W(C) \cap V$.

It follows from $\emptyset \neq A \cap U \subset A \cap W(C)$ that the set A meets $W(C)$ and hence is contained in $W(C)$ and is disjoint with $F_C = [X \setminus \bigcup \mathcal{V}(C)]_{\mathcal{C}}$. Hence

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Next, take any set $U' \in \mathcal{U}_n(C)$ and find a set $V' \in \mathcal{V}(C)$ with $U' = W(C) \cap V'$. The (in)equality $A \cap W(C) \cap V = A \cap U \neq \emptyset$ and the definition of the set $W(C) \supset A$ implies that A intersects $V' \in \mathcal{V}(C)$ and hence intersects $U' = W(C) \cap V'$. This completes the proof of Claim.

Proof of the Claim (continuation)

Let us show that the cover $\mathcal{U}_n(C)$ satisfies the condition

(ii) if a set $A \in \mathcal{C}$ meets some set $U \in \mathcal{U}_n(C)$, then $A \subset \bigcup \mathcal{U}_n(C)$ and A meets each set $U' \in \mathcal{U}_n(C)$.

Assume that a set $A \in \mathcal{C}$ meets some set $U \in \mathcal{U}_n(C)$.

First we show that $A \subset \bigcup \mathcal{U}_n(C)$.

Find a set $V \in \mathcal{V}(C)$ such that $U = W(C) \cap V$.

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Assume that a set $A \in \mathcal{C}$ meets some set $U \in \mathcal{U}_n(C)$.

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Proof of Theorem (cotinuation)

Given two points $x, y \in X$ let

$$\delta(x,y) := \inf\left\{\tfrac{1}{2^n} : \exists C \in \mathcal{C} \text{ and } U \in \mathcal{U}_n(C) \text{ such that } (x,y) \in U\right\}.$$

Adjust the function δ to a pseudometric d letting

$$d(x,y) = \inf \sum_{i=1}^{m} \delta(x_{i-1}, x_i)$$

where the infimum is taken over all sequences $x = x_0, \dots, x_m = y$.

The condition (i) of Claim implies that $\rho(x,y) \le \delta(x,y)$ and hence $\rho(x,y) \le d(x,y)$ for any $x, y \in X$. So, the pseudometric d is a metric on X such that the identity map $(X,d) \to (X,\rho)$ is continuous. To see that this map is a homeomorphism, take any point $x \in X$ and $\varepsilon > 0$. Find $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \varepsilon$ and choose a set $C \in \mathcal{C}$ with $x \in C$ and a set $U \in \mathcal{U}_n(C)$ with $x \in U$.

Then for any $y \in U$ we get $d(y,x) \le \delta(x,y) \le \frac{1}{2^n} < \varepsilon$, which means that the map $X \to (X,d)$ is continuous.

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Proof of Theorem (continuation)

We claim that the metric d is $\mathcal{C}$-parallel.

Given two distinct compact sets $A, B \in \mathcal{C}$, we need to show that $d(a, B) = d(A, B) = d(A, b)$ for any $a \in A$, $b \in B$.

Assuming that this inequality is not true, we conclude that either

$$d(a, B) > d(A, B) > 0 \text{ or } d(A, b) > d(A, B) > 0$$

for some $a \in A$ and $b \in B$.

First assume that $d(a, B) > d(A, B)$ for some $a \in A$. Choose points $a' \in A$, $b' \in B'$ such that $d(a', b') = d(A, B) < d(a, B)$.

By the definition of the distance $d(a', b') < d(a, B)$, there exists a chain $a' = x'_0, x'_1, \dots, x'_m = b'$ such that

$$\sum_{i=1}^{m} \delta(x'_{i-1}, x'_i) < d(a, B).$$

We can assume that the points $x'_0, \dots, x'_m$ are pairwise distinct, so for every $i \le m$ there exist $n_i \ge 0$ such that $\delta(x'_{i-1}, x'_i) = \frac{1}{2^{n_i}}$ and hence $x'_{i-1}, x'_i \in U'_i$ for some $C_i \in \mathcal{C}$ and $U'_i \in \mathcal{U}_{n_i}(C_i)$. For every $i \le m$ let $A_i \in \mathcal{C}$ be the unique set containing x'_i. Then $A_0 = A$ and $A_m = B$.

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First assume that $d(a, B) > d(A, B)$ for some $a \in A$. Choose points $a' \in A$, $b' \in B'$ such that $d(a', b') = d(A, B) < d(a, B)$.

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Proof of Theorem (continuation)

We claim that the metric d is $\mathcal{C}$-parallel.

Given two distinct compact sets $A, B \in \mathcal{C}$, we need to show that $d(a, B) = d(A, B) = d(A, b)$ for any $a \in A$, $b \in B$.

Assuming that this inequality is not true, we conclude that either

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Proof of Theorem (the end)

Using the condition

(ii) if a set $A \in \mathcal{C}$ meets some set $U \in \mathcal{U}_n(C)$, then $A \subset \bigcup \mathcal{U}_n(C)$ and A meets each set $U' \in \mathcal{U}_n(C)$.

of Claim, we can inductively construct a sequence of points $a = x_0, x_1, \dots, x_m$ such that for every positive $i \le m$ the point x_i belongs to A_i and the points x_{i-1}, x_i belong to some set $U_i \in \mathcal{U}_{n_i}(C_i)$. Then $x_m \in A_m = B$.

The chain $a = x_0, x_1, \dots, x_m$ witnesses that

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By analogy we can prove that the case $d(A,B) < d(A,b)$ leads to a contradiction.

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