A non-Hurewicz set of reals of size $\mathfrak{d}$ with all its powers Menger

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(1). Basic definitions and background

(2). Main results

(3). Application
Definition (1924)

A set $X$ has the *Menger* property if for each sequence
\[ \{U_n : n \in \mathbb{N}\} \] of open covers of $X$ there exist finite subsets
\[ V_n \subset U_n, \; n \in \mathbb{N}, \] such that the collection \[ \bigcup\{V_n : n \in \mathbb{N}\} \] is a cover of $X$.

Fact

\[ \sigma \text{-compact} \implies \textit{Menger property} \implies \textit{Lindelöf}. \]

Example

$\mathbb{N}^\mathbb{N}$ is a Lindelöf space, and does not satisfy Menger property.
Menger’s Conjecture: Each separable metric Menger space is $\sigma$-compact.

Recall that $X \subset \mathbb{R}$ is a Lusin set if it is uncountable, and for every meager set (a union of countably many nowhere dense sets) $A \subset \mathbb{R}$, $X \cap A$ is countable.

**Fact (Mahlo, Lusin)**

(CH) There exists a Lusin set in $\mathbb{R}$.

**Theorem (Hurewicz, 1925)**

Every Lusin set has Menger property, and is not $\sigma$-compact.
Notations:

For $a, b \in \mathbb{N}^\mathbb{N}$, denote $a \leq^* b$ if $a(n) \leq b(n)$ for all but finitely many $n$. $a \nless^* b$ is denoted by $b <^\infty a$.

$\mathcal{U}$ is an open cover of $X$ if for every $U \in \mathcal{U}$, $U$ is an open subset of $X$, and $\bigcup \mathcal{U} = X$, and $X \notin \mathcal{U}$.

$\mathcal{U}$ is a $\gamma$-cover of $X$ if for each $x \in X$, $x$ is contained in all but finitely many elements of $\mathcal{U}$.

$\mathcal{U}$ is a $\omega$-cover of $X$ if for each finite $F \subset X$, there exist a $U \in \mathcal{U}$ such that $F \subset U$. 

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Definition (1925)

A space has *Hurewicz* property if for each sequence \( \{U_n : n \in \mathbb{N}\} \) of open covers of \( X \) there exist finite subsets \( V_n \subset U_n, n \in \mathbb{N}, \) such that the collection \( \{\bigcup V_n : n \in \mathbb{N}\} \) is a \( \gamma \)-cover of \( X \).

**Fact:** \( \sigma \)-compact \( \Rightarrow \) *Hurewicz* property \( \Rightarrow \) *Menger* property.

**Hurewicz’s Conjecture:** Each separable metric Hurewicz space is \( \sigma \)-compact.
A set $X \subset \mathbb{R}$ is called Sierpiński set if it is uncountable and for every Lebesgue measure zero set $M \subset \mathbb{R}$, $M \cap X$ is countable.

**Fact (Sierpiński)**

(CH) There exists a Sierpiński set in $\mathbb{R}$.

**Theorem (folklore)**

Every Sierpiński set has Hurewicz property, and is not $\sigma$-compact.
Notations: $\mathbb{N}^\uparrow \mathbb{N}$ stands for all increasing members of $\mathbb{N}^\mathbb{N}$.

$[\mathbb{N}]^{\mathbb{N}}$ stands for all infinite subsets of $\mathbb{N}$.

$Fin$ stands for all finite subsets of $\mathbb{N}$.

For $a \in [\mathbb{N}]^{\mathbb{N}}$, $y \in \mathbb{N}^\uparrow \mathbb{N}$, denote $y/a = \{n : a \cap [y(n), y(n+1)) \neq \emptyset\}$.
The distinction of $\sigma$-compact and Hurewicz property, the distinction of $\sigma$-compact and Menger property in $ZFC$

A $\mathfrak{d}$-scale is a dominating family $\{s_\alpha : \alpha < \mathfrak{d}\} \subseteq [\mathbb{N}]^\mathbb{N}$ such that for all $\alpha < \beta < \mathfrak{d}$, $s_\beta \not\leq^* s_\alpha$; A $\mathfrak{b}$-scale is a unbounded family $\{b_\alpha : \alpha < \mathfrak{b}\} \subseteq [\mathbb{N}]^\mathbb{N}$ such that for all $\alpha < \beta < \mathfrak{b}$, $s_\alpha \leq^* s_\beta$.

**Theorem (Bartoszyński-Tsaban, 2006, Proceedings of AMS)**

1. For each $\mathfrak{d}$-scale $S$, $S \cup Fin$ has Menger property and is not $\sigma$-compact.
2. For each $\mathfrak{b}$-scale $S$, $S \cup Fin$ has Hurewicz property and is not $\sigma$-compact.
Definition

Let $P$ be a topological property, let

$$non(P) = \min\{|X|, X \text{ not has property } P\}.$$ 

Lemma (Hurewicz, 1927, Fund. Math)

1. $non(Menger) = 0$.
2. $non(Hurewicz) = b$. 

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Theorem (J. Chaber, R. Pol, 2002)

$(\mathfrak{b} = \mathfrak{d})$ There is a space with size $\mathfrak{d}$ such that $X$ has Menger property and does not satisfies Hurewicz property.

Theorem (Tsaban, Zdomskyy, 2008, JEMS)

There is, in ZFC, a set of reals of cardinality $\mathfrak{d}$ that is Menger but not Hurewicz.
Notations: $\mathcal{O}$ denotes all open covers of $X$.

$\Omega$ denotes the collection of all $\omega$-covers of $X$.

$\Gamma$ denotes the collection of all $\gamma$-covers of $X$.

Definition $S_{fin}(\Omega, \Omega)$: For any sequence $\{U_n : n \in \mathbb{N}\} \subset \Omega$, there is $V_n \in [U_n]^{<\mathbb{N}}$ such that $\bigcup\{V_n : n \in \mathbb{N}\} \in \Omega$. 
Notation: \( U_{fin}(\mathcal{O}, \Gamma) = \text{Hurewicz}; \ S_{fin}(\Omega, \Omega) = \text{Menger}. \)

Theorem (W. Just, A. Miller, M. Scheepers, P. Szeptycki, 1996, Topol Appl)

Let \( X \) be a separable metrizable space. Then \( X \) is \( S_{fin}(\Omega, \Omega) \) if and only if every finite power of \( X \) has Menger property.
How are $S_{fin}(\Omega, \Omega)$ and $U_{fin}(\mathcal{O}, \Gamma)$ different.

**Theorem (Chaber-Pol, 2002)**

(b = ℵ) *There is a space with size ℵ such that*

$X \in S_{fin}(\Omega, \Omega) \setminus U_{fin}(\mathcal{O}, \Gamma)$.

**Theorem (Tsaban, Zdomskyy, 2008, JEMS)**

*If ℵ is regular, then there is a non-Hurewicz set of reals of size ℵ with all its powers Menger.*
Question (Tsaban, 2009, Contemporary Math)

Is there, in $ZFC$, a space with size $\mathfrak{d}$ such that $X \in S_{fin}(\Omega, \Omega) \setminus U_{fin}(\emptyset, \Gamma)$. 

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Main results

**Definition**

$X \subseteq \mathbb{N}^\mathbb{N}$ is called $\kappa$-	extit{unbounded} if for any $g \in \mathbb{N}^\mathbb{N}$,
\[|\{x \in X : x \leq^* g\}| < \kappa.\]

**Notion:** $X \subseteq [\mathbb{N}]^\mathbb{N}$ is called $\kappa$-	extit{unbounded} if the enumeration of its elements in $\mathbb{N}^\mathbb{N}$ is $\kappa$-	extit{unbounded}.

**Definition**

A set $X \subseteq [\mathbb{N}]^\mathbb{N}$ with $|X| \geq \kappa$ is called $\kappa$-	extit{concentrated} if it contains a countable set $D$ such that $|X \setminus U| < \kappa$ for any open $U$ containing $D$. 
Main results

Lemma (folklore)

*Each $\vartheta$-concentrated space has Menger property.*

The relation between the $\kappa$-concentrated sets and the $\kappa$-unbounded sets is as follows.

Lemma (P. Szewczak, B. Tsaban, 2017, Ann Pure and Appl Logic)

*Let $\kappa$ be an infinite cardinal number and $X \subseteq [\mathbb{N}]^\mathbb{N}$ with $|X| \geq \kappa$. The set $X$ is $\kappa$-unbounded if and only if the set $X \cup \text{Fin}$ is $\kappa$-concentrated on Fin.*
Main results

Definition

For any \( g \in \mathbb{N}^\mathbb{N} \), define \( \tilde{g} \) as follows.

\[
\tilde{g}(0) = g(0); \quad \tilde{g}(n + 1) = \tilde{g}(n) + g(\tilde{g}(n)).
\]

Notation: \( \chi_s \) stands for the characteristic function of \( s \), \( s \in P(\mathbb{N}) \).

Definition

For any \( s \in [\mathbb{N}]^\mathbb{N} \), \( f_s(n) = k \) if the length of 0s between the \( n \)-th '1' and the \( (n + 1) \)-th '1' of \( \chi_s \) is equal to \( k \).

Well-known Fact: The map \( s \to f_s \) is a homeomorphism.
Main results

Lemma (J. He, J. Yu, S. Zhang)

Let $a \in \mathbb{N}^\mathbb{N}$, $g \in \mathbb{N}^{\uparrow \mathbb{N}}$ and $a$ omits an interval $I$ with at least two points from $\tilde{g}$. Then there exists a $k \in \mathbb{N}$ such that $g(k) < f_a(k)$.

Proof.

Assume that $i < j$ are consecutive elements of $I \cap \tilde{g}$.

Put $k = |a \cap \text{min} I|.$

Then $g(k) \leq g(i) = j - i < |I| < f_a(k).$
Main results

**Lemma (folklore)**

Let $Y \subset [\mathbb{N}]^\mathbb{N}$ with $|Y| < \mathfrak{d}$. Then there is a $b \in \mathbb{N}^\mathbb{N}$ such that for each $y \in Y$, the set $\{n : |y \cap (b(n), b(n+1)| \geq 2\}$ is infinite.

**Lemma (J. He, J. Yu, S. Zhang)**

For any $Y \subseteq \mathbb{N}^\mathbb{N}$ with $|Y| < \mathfrak{d}$, for any $a \in \mathbb{N}^\mathbb{N}$. $\exists I \in [\mathbb{N}]^\mathbb{N}$ such that $c = \bigcup_{n \in I} [a(n), a(n+1))$ satisfies that $Y < \infty f_c$.

**Corollary (J. He, J. Yu, S. Zhang)**

For any $Y \subseteq \mathbb{N}^\mathbb{N}$ with $|Y| < \mathfrak{d}$, for any $g \in \mathbb{N}^\mathbb{N}$. $\exists s \in [\mathbb{N}]^\mathbb{N}$ such that $Y < \infty f_s$ and $g < \infty f_{sc}$.
The following is our main result which answers the Tsaban’s question mentioned above:

**Theorem (J. He, J. Yu, S. Zhang)**

In ZFC, there exists a space $X \subset P(\mathbb{N})$ such that

1. $|X| = \mathfrak{d}$
2. $X$ is not $U_{\text{fin}}(\mathcal{O}, \Gamma)$.
3. $X^n$ has $S_{\text{fin}}(\mathcal{O}, \mathcal{O})$ for every $n$. 

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Proof.

Fix a dominating set \( \{d_\alpha : \alpha < \mathfrak{d}\} \subseteq [\mathbb{N}]^\mathbb{N} \) with size \( \mathfrak{d} \). Constructing inductively a family \( \{f_{s_\gamma} : \gamma < \mathfrak{d}\} \subseteq \mathbb{N}^\mathbb{N} \) such that the set

\[
X = \{s_\gamma : \gamma < \mathfrak{d}\} \cup \text{Fin}
\]

has Menger property but no Hurewicz property.

In step 0. We can choose a \( f_{s_0} \) such that the set \( f_{s_0} \in \mathbb{N}^\mathbb{N} \) and

\[
\{d_0\} \prec f_{s_0}, f_{s_0}.
\]

In step \( \alpha < \mathfrak{d} \). Note that \( |\{d_\beta : \beta < \alpha\}| < \mathfrak{d} \), there exists a \( f_{s_\alpha} \) such that \( f_{s_\alpha} \in \mathbb{N}^\mathbb{N} \) and

\[
\{d_\beta : \beta < \alpha\} \prec f_{s_\alpha}, f_{s_\alpha}.
\]
The sketch of the proof

Proof.

It is not hard to see that \( \{ f_{s, \gamma} : \gamma < d \} \) is \( d \)-unbounded, so is the set \( \{ s_\gamma : \gamma < d \} \)

Recall that the complement function \( \tau : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N}) \)

\[ \tau(A) = \mathbb{N} \setminus A \]

is a homeomorphism. We define the following map \( \psi : P(\mathbb{N}) \to \mathbb{N}^\mathbb{N} \)

such that

\[ \psi : s \to s^c \to f_{s^c}. \]

\( \psi \) is a continuous function, and \( \psi(X \cup Fin) \) contains a unbounded family \( \{ f_{s^c, \gamma} : \gamma < d \} \).

Thus, \( X \cup Fin \) is not a Hurewicz space.
Fact

\((X \cup \text{Fin})^2 \text{ is Menger.}\)

Proof of the Fact

**Proof.**

Note that

\[
(X \cup \text{Fin})^2 = (X \times X) \cup (\text{Fin} \times \text{Fin}) \cup ((X \cup \text{Fin}) \times \text{Fin}) \cup (\text{Fin} \times (X \cup \text{Fin})).
\]

Define \(f: P(\mathbb{N}) \to P(\mathbb{N}) \times P(\mathbb{N}).\)

such that

\[
f(x) = (x_0, x_1).
\]

Where \(x_0 = \{x(2n) : n < \mathbb{N}\}, \ x_1 = \{x(2n + 1) : n < \mathbb{N}\}.
\]
Proof.

Then \( f \) is continuous and

\[
f((X \bigoplus X) \cup \text{Fin}) = (X \times X) \cup (\text{Fin} \times \text{Fin}).
\]

We need to show that

\[(X \bigoplus X) \cup \text{Fin} \text{ is } \mathfrak{d}\text{-concentrated on } \text{Fin}.
\]

Notice that \( X \bigoplus X \) homeomorphic to \( \{f_{s\alpha} \bigoplus f_{s\beta} : \alpha, \beta < \mathfrak{d}\} \).

we just need to show that

\[
\{f_{s\alpha} \bigoplus f_{s\beta} : \alpha, \beta < \mathfrak{d}\} \text{ is } \mathfrak{d}\text{-unbounded in } \mathbb{N}^\mathbb{N}.
\]
Proof.

Let $b$ be any element of $\mathbb{N}^\mathbb{N}$ and $b = b_0 \oplus b_1$.

where $b_0 = \{b(2n) : n \in \mathbb{N}\}$ and $b_1 = \{b(2n + 1) : n \in \mathbb{N}\}$.

It is easy to see that

$$f_{s\alpha} \oplus f_{s\beta} \leq^* b \text{ if and only if } f_{s\alpha} \leq^* b_0 \text{ and } f_{s\beta} \leq^* b_1.$$ 

Then we have that

$$\{f_{s\alpha} \oplus f_{s\beta} : f_{s\alpha} \oplus f_{s\beta} \leq^* b, \alpha, \beta < \mathfrak{d}\}$$

$$= \{f_{s\alpha} : f_{s\alpha} \leq^* b_0, \alpha < \mathfrak{d}\} \cap \{f_{s\beta} : f_{s\alpha} \leq^* b_1, \beta < \mathfrak{d}\}.$$
Proof.

Note that

\[ \left| \left\{ f_{s\alpha} : f_{s\alpha} \leq^* b_0, \alpha < \mathfrak{d} \right\} \right| < \mathfrak{d} ; \]

\[ \left| \left\{ f_{s\beta} : f_{s\alpha} \leq^* b_1, \beta < \mathfrak{d} \right\} \right| < \mathfrak{d} ; \]

Thus,

\[ \left| \left\{ f_{s\alpha} \oplus f_{s\beta} : f_{s\alpha} \oplus f_{s\beta} \leq^* b, \alpha, \beta < \mathfrak{d} \right\} \right| < \mathfrak{d}. \]

Then we have completed the proof of the Fact above.
Fact

For every \( n > 2 \), \( (X \cup Fin)^n \) satisfies Menger property.

Proof.

For each \( n \), Notice that \( X \times Fin \times X \) is homeomorphic to \( X \times X \times Fin \). We have that

\[
(X \cup Fin)^k = \bigcup_{1 \leq k \leq n} C_n^k(X^k \times Fin^{n-k})
\]

\[
= \bigcup_{1 \leq k \leq n} C_n^k((X^k \times Fin^{n-k}) \cup Fin^n)
\]

\[
= \bigcup_{1 \leq k \leq n} C_n^k((X^k \cup Fin^k) \times Fin^{n-k})
\]

It is enough to show that \( X^k \cup Fin^k \) is Menger for each \( 1 \leq k \leq n \), and the proof is similar to that of the case \( n = 2 \) above.
Proof.

Define a function

\[ P(\mathbb{N}) \to (P(\mathbb{N}))^k. \]

such that

\[ f(x) = \langle x(0), x(1), \ldots x(n - 1) \rangle. \]

Where \( x(i) = \langle x(k) : k \equiv i \mod(n) \rangle \) for each \( 1 \leq i \leq n \).

It is clearly that \( f \) is continuous. Moreover.

\[ f(\bigoplus_{1 \leq i \leq k} X_i \cup Fin) = X^k \cup Fin^k. \]

Where \( X = X_i \) for each \( i \leq k \).
Proof.

**Observation:** $\bigoplus_{1 \leq i \leq k} X_i \cup Fin$ is $\mathcal{d}$-concentrated on $Fin$.

To see this, recall that

$$\bigoplus_{1 \leq i \leq k} X_i \approx \{\bigoplus_{j \in F} f_{s_j} : f_{s_j} \in X, F \in [\mathcal{d}]^k\}.$$ 

Let $b$ be any element of $\mathbb{N}^{\mathbb{N}}$ and $b = b_0 \oplus b_1 \oplus \cdots \oplus b(n-1)$. Where

$$b_i = \{b(k) : k \equiv i \mod(n)\}.$$ 

It is easy to see that

$$\bigoplus_{j \in F} f_{s_j} \leq^* b \text{ if and only if } f_{s_i} \leq^* b_i \text{ for each } i < n.$$
The sketch of the proof

Proof.

Then we have that

$$\left\{ \bigoplus_{j \in F} f_{s_j} : \bigoplus_{j \in F} f_{s_j} \leq^* b \right\} = \bigcap_{i \leq n-1} \left\{ f_{s_\alpha} : f_{s_\alpha} \leq^* b(i), \alpha < \varpi \right\}.$$

Note that the space $X$ is $\varpi$-unbounded, so for all $i \leq n - 1$, both the cardinalities of $\left\{ f_{s_\alpha} : f_{s_\alpha} \leq^* b(i), \alpha < \varpi \right\}$ are less than $\varpi$. Therefore,

$$|\left\{ \bigoplus_{j \in F} f_{s_j} : \bigoplus_{j \in F} f_{s_j} \leq^* b, \bigoplus_{j \in F} f_{s_j} \in \bigoplus_{1 \leq i \leq k} X_i \right\}| < \varpi.$$

Thus, $\bigoplus_{1 \leq i \leq k} X_i$ is $\varpi$-unbounded, and then $\bigoplus_{1 \leq i \leq k} X_i \cup Fin$ is $\varpi$-concentrated on $Fin$ as desired.
Application

Definition

A semi-filter on an infinite countable $X$ is a non-empty family $\mathcal{F} \subseteq [X]^\mathbb{N}$ containing all almost-supersets of its elements. A filter is a semi-filter closed under finite intersections.

Definition

For a filter $\mathcal{F}$ on $X$, $\mathcal{B} \subseteq \mathcal{F}$ is called a base of $\mathcal{F}$ if for each $F \in \mathcal{F}$, there is a $B \in \mathcal{B}$ with $B \subseteq F$. The character of a filter is the minimal cardinality of its bases.
How are different among the property $S_{fin}(\mathcal{O}, \mathcal{O})$, $U_{fin}(\mathcal{O}, \Gamma)$, $U_{fin}(\mathcal{O}, \Omega)$ and $S_{fin}(\Omega, \Omega)$ in the realm of filters?

Recall that

**Proposition (D. Chodounsky, D. Repovš, L. Zdomskky, 2014, J.S.L)**

*Let $\mathcal{F}$ be a filter on $\mathbb{N}$. Then $\mathcal{F}$ is Menger (Hurewicz) then for all $0 < n < \mathbb{N}$, $\mathcal{F}^n$ is Menger (Hurewicz).*

In the realm of filters,

\[ U_{fin}(\mathcal{O}, \Gamma) \rightarrow U_{fin}(\mathcal{O}, \Omega) \leftarrow S_{fin}(\mathcal{O}, \mathcal{O}) \]

\[ S_{fin}(\Omega, \Omega) \]
Lemma (R.Hernández-Gutiérrez, P. Szeptycki, 2015, Comm Math Universitatis Carolinae)

(1) \( b \) is the minimal character of a filter that is not Hurewicz.

(2) \( \delta \) is the minimal character of a filter that is not Menger.

Lemma (R.Hernández-Gutiérrez, P. Szeptycki, 2015, Comm Math Universitatis Carolinae)

\( (b = \delta) \) There exist a Menger filter of character \( \delta \) that is not Hurewicz.
Question (R. Hernández-Gutiérrez, P. Szeptycki, 2015, Comm Math Universitatis Carolinae)

Is there, in ZFC, a Menger filter of character $\mathfrak{d}$ that is not Hurewicz?
Given a set $X \subset 2^\mathbb{N}$, let $\mathcal{I}_X$ be the ideal generated by finite union of branches $C_x = \{ x \mid n : n \in \mathbb{N} \}$ where $x \in X$ and $\mathcal{F}_X$ be the dual filter.

Lemma (R. Hernández-Gutiérrez, P. Szeptycki, 2015, Comm Math Universitatis Carolinae)

Let $X \subset 2^\mathbb{N}$. Then

1. every finite power of $X$ is Menger if and only if $\mathcal{F}_X$ is Menger.
2. every finite power of $X$ is Hurewicz if and only if $\mathcal{F}_X$ is Hurewicz.

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# Application

**Theorem (J. He, J. Yu, S. Zhang)**

There exists, in ZFC, a Menger filter with character \( \mathfrak{c} \) that is not Hurewicz.

**Proof.**

Let \( X \) be a finite power Menger space with size \( \mathfrak{c} \) that is not Hurewicz (the existence was proved in the previous paragraph).

Note that the set \( \{(C_x)^c : x \in X\} \) is the filter base for \( \mathcal{F}_X \), and \(|\{(C_x)^c : x \in X\}| = \mathfrak{c} \).

By the lemma above, the filter \( \mathcal{F}_X \) meets the bill. \( \square \)
Thank you!