A non-Hurewicz set of reals of size $\mathfrak{d}$ with all its powers Menger

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Outline

(1). Basic definitions and background

(2). Main results

(3). Application

Basic definitions and background

Definition (1924)

A set X has the $Menger$ property if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X there exist finite subsets $\mathcal{V}_n \subset \mathcal{U}_n$, $n \in \mathbb{N}$, such that the collection $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a cover of X.

Fact

σ-compact $\Rightarrow$ $Menger$ property $\Rightarrow$ Lindelöff.

Example

$\mathbb{N}^{\mathbb{N}}$ is a Lindelöff space, and does not satisfy Menger property.

Basic definitions and background

Menger's Conjecture: Each separable metric Menger space is σ-compact.

Recall that $X \subset \mathbb{R}$ is a $Lusin$ set if it is uncountable, and for every meager set (a union of countably many nowhere dense sets) $A \subset \mathbb{R}$, $X \cap A$ is countable.

Fact (Mahlo, Lusin)

(CH) *There exists a Lusin set in* $\mathbb{R}$.

Theorem (Hurewicz, 1925)

Every Lusin set has Menger property, and is not σ*-compact.*

Notations:

For $a,\ b \in \mathbb{N}^{\mathbb{N}}$, denote $a \leq^* b$ if $a(n) \leq b(n)$ for all but finitely many n. $a \not\leq^* b$ is denoted by $b <^\infty a$.

$\mathcal{U}$ is an open cover of X if for every $U \in \mathcal{U}$, U is an open subset of X, and $\bigcup \mathcal{U} = X$, and $X \notin \mathcal{U}$.

$\mathcal{U}$ is a γ-cover of X if for each $x \in X$, x is contained in all but finitely many elements of $\mathcal{U}$.

$\mathcal{U}$ is a ω-cover of X if for each finite $F \subset X$, there exist a $U \in \mathcal{U}$ such that $F \subset U$.

Definition (1925)

A space has *Hurewicz* property if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X there exist finite subsets $\mathcal{V}_n \subset \mathcal{U}_n$, $n \in \mathbb{N}$, such that the collection $\{\cup\mathcal{V}_n : n \in \mathbb{N}\}$ is a γ-cover of X

Fact: σ-compact $\Rightarrow$ *Hurewicz* property $\Rightarrow$ *Menger* property.

Hurewicz's Conjecture: Each separable metric Hurewicz space is σ-compact.

Basic definitions and background

A set $X \subset \mathbb{R}$ is called Sierpiński set if it is uncountable and for every Lebesgue measure zero set $M \subset \mathbb{R}$, $M \cap X$ is countable.

Fact (Sierpiński)

(CH) *There exists a Sierpiński set in* $\mathbb{R}$.

Theorem (folklore)

Every Sierpiński set has Hurewicz property, and is not σ*-compact.*

Notations: $\mathbb{N}^{\uparrow\mathbb{N}}$ stands for all increasing members of $\mathbb{N}^{\mathbb{N}}$.

$[\mathbb{N}]^{\mathbb{N}}$ stands for all infinite subsets of $\mathbb{N}$.

Fin stands for all finite subsets of $\mathbb{N}$.

For $a \in [\mathbb{N}]^{\mathbb{N}}$, $y \in \mathbb{N}^{\uparrow\mathbb{N}}$, denote $y/a = \{n : a \cap [y(n), y(n+1)) \neq \emptyset\}$.

Basic definitions and background

The distinction of σ-compact and Hurewicz property, the distinction of σ-compact and Menger property in ZFC

A $\mathfrak{d}$-scale is a dominating family $\{s_\alpha : \alpha < \mathfrak{d}\} \subseteq [\mathbb{N}]^{\mathbb{N}}$ such that for all $\alpha < \beta < \mathfrak{d}$, $s_\beta \nleq^* s_\alpha$; A $\mathfrak{b}$-scale is a unbounded family $\{b_\alpha : \alpha < \mathfrak{b}\} \subseteq [\mathbb{N}]^{\mathbb{N}}$ such that for all $\alpha < \beta < \mathfrak{d}$, $s_\alpha \leq^* s_\beta$.

Theorem (Bartoszyński-Tsaban, 2006, Proceedings of AMS)

(1) *For each $\mathfrak{d}$-scale S, $S \bigcup Fin$ has Menger property and is not σ-compact.*

(2) *For each $\mathfrak{b}$-scale S, $S \bigcup Fin$ has Hurewicz property and is not σ-compact.*

Definition

Let P be a topological property, let
$non(P) = min\{|X|, X\ not\ has\ property\ P\}$.

Lemma (Hurewicz, 1927, Fund. Math)

(1) $non(Menger) = \mathfrak{d}$.
(2) $non(Hurewicz) = \mathfrak{b}$.

Basic definitions and background

Theorem (J. Chaber, R. Pol, 2002)

($\mathfrak{b} = \mathfrak{d}$) There is a space with size $\mathfrak{d}$ such that X has Menger property and does not satisfies Hurewicz property.

Theorem (Tsaban, Zdomskyy, 2008, JEMS)

There is, in ZFC, a set of reals of cardinality $\mathfrak{d}$ that is $Menger$ but not Hurewicz.

Basic definitions and background

Notations: $\mathcal{O}$ denotes all open covers of X.

Ω denotes the collection of all ω-covers of X.

Γ denotes the collection of all γ-covers of X.

Definition

$S_{fin}(\Omega, \Omega)$: For any sequence $\{\mathcal{U}_n : n \in \mathbb{N}\} \subset \Omega$, there is $\mathcal{V}_n \in [\mathcal{U}_n]^{<\mathbb{N}}$ such that $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\} \in \Omega$.

Basic definitions and background

Notation: $U_{fin}(\mathcal{O}, \Gamma)$=Hurewicz; $S_{fin}(\Omega, \Omega)$=Menger.

Theorem (W. Just, A. Miller, M. Scheepers, P. Szeptycki, 1996, Topol Appl)

Let X be a separable metrizable space. Then X is $S_{fin}(\Omega, \Omega)$ if and only if every finite power of X has Menger property.

How are $S_{fin}(\Omega, \Omega)$ and $U_{fin}(\mathcal{O}, \Gamma)$ different.

Theorem (Chaber-Pol, 2002)

($\mathfrak{b} = \mathfrak{d}$) There is a space with size $\mathfrak{d}$ such that $X \in S_{fin}(\Omega, \Omega) \setminus U_{fin}(\mathcal{O}, \Gamma)$.

Theorem (Tsaban, Zdomskyy, 2008, JEMS)

If $\mathfrak{d}$ is regular, then there is a non-Hurewicz set of reals of size $\mathfrak{d}$ with all its powers Menger.

Question (Tsaban, 2009, Contemporary Math)

Is there, in ZFC, a space with size $\mathfrak{d}$ such that $X \in S_{fin}(\Omega, \Omega) \setminus U_{fin}(\mathcal{O}, \Gamma)$.

Main results

Definition

$X \subseteq \mathbb{N}^{\mathbb{N}}$ is called *κ-unbounded* if for any $g \in \mathbb{N}^{\mathbb{N}}$, $|\{x \in X : x \leq^* g\}| < \kappa$.

Notion: $X \subseteq [\mathbb{N}]^{\mathbb{N}}$ is called *κ-unbounded* if the enumeration of its elements in $\mathbb{N}^{\mathbb{N}}$ is *κ-unbounded*.

Definition

A set $X \subseteq [\mathbb{N}]^{\mathbb{N}}$ with $|X| \geq \kappa$ is called *κ-concentrated* if it contains a countable set D such that $|X \backslash U| < \kappa$ for any open U containing D.

Main results

Lemma (folklore)

Each $\mathfrak{d}$-concentrated space has Menger property.

The relation between the κ-$concentrated$ sets and the κ-$unbounded$ sets is as follows.

Lemma (P. Szewczak, B. Tsaban, 2017, Ann Pure and Appl Logic)

Let κ be a infinite cardinal number and $X \subseteq [\mathbb{N}]^{\mathbb{N}}$ with $|X| \geq \kappa$. The set X is κ-$unbounded$ if and only if the set $X \cup Fin$ is κ-$concentrated$ on Fin.

Main results

Definition

For any $g \in \mathbb{N}^{\mathbb{N}}$, define $\tilde{g}$ as follows.

$$\tilde{g}(0) = g(0);\ \tilde{g}(n+1) = \tilde{g}(n) + g(\tilde{g}(n)).$$

Notation:χ_s stands for the characteristic function of s, $s \in P(\mathbb{N})$.

Definition

For any $s \in [\mathbb{N}]^{\mathbb{N}}$, $f_s(n) = k$ if the length of 0s between the n-th '1' and the $(n+1)$-th '1' of χ_s is equal to k.

Well-known Fact: The map $s \to f_s$ is a homeomorphism.

Main results

Lemma (J. He, J. Yu, S. Zhang)

Let $a \in [\mathbb{N}]^{\mathbb{N}}$, $g \in \mathbb{N}^{\uparrow\mathbb{N}}$ and a omits an interval I with at least two points from $\tilde{g}$. Then there exists a $k \in \mathbb{N}$ such that $g(k) < f_a(k)$.

Proof.

Assume that $i < j$ are consecutive elements of $I \cap \tilde{g}$.

Put $k = |a \cap minI|$.

Then $g(k) \leq g(i) = j - i < |I| < f_a(k)$.

□

Main results

Lemma (folklore)

Let $Y \subset [\mathbb{N}]^{\mathbb{N}}$ *with* $|Y| < \mathfrak{d}$. *Then there is a* $b \in \mathbb{N}^{\uparrow\mathbb{N}}$ *such that for each* $y \in Y$, *the set* $\{n : |y \cap (b(n), b(n+1)| \geq 2\}$ *is infinite.*

Lemma (J. He, J. Yu, S. Zhang)

For any $Y \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ *with* $|Y| < \mathfrak{d}$, *for any* $a \in \mathbb{N}^{\uparrow\mathbb{N}}$. $\exists I \in [\mathbb{N}]^{\mathbb{N}}$ *such that* $c = \bigcup_{n \in I}[a(n), a(n+1))$ *satisfies that* $Y <^{\infty} f_c$.

Corollary (J. He, J. Yu, S. Zhang)

For any $Y \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ *with* $|Y| < \mathfrak{d}$, *for any* $g \in \mathbb{N}^{\uparrow\mathbb{N}}$. $\exists s \in [\mathbb{N}]^{\mathbb{N}}$ *such that* $Y <^{\infty} f_s$ *and* $g <^{\infty} f_{s^c}$.

Main results

The following is our main result which answers the Tsaban's question mentioned above:

Theorem (J. He, J. Yu, S. Zhang)

In ZFC, there exists a space $X \subset P(\mathbb{N})$ such that

(1). $|X| = \mathfrak{d}$

(2). X *is not* $U_{fin}(\mathcal{O}, \Gamma)$.

(3). X^n *has* $S_{fin}(\mathcal{O}, \mathcal{O})$ *for every* n.

The sketch of the proof

Proof.

Fix a dominating set $\{d_\alpha : \alpha < \mathfrak{d}\} \subseteq [\mathbb{N}]^{\mathbb{N}}$ with size $\mathfrak{d}$. Constructing inductively a family $\{f_{s_\gamma} : \gamma < \mathfrak{d}\} \subseteq \mathbb{N}^{\mathbb{N}}$ such that the set

$$X = \{s_\gamma : \gamma < \mathfrak{d}\} \bigcup Fin$$

has Menger property but no Hurewicz property.

In step 0. We can choose a f_{s_0} such that the set $f_{s_0^c} \in \mathbb{N}^{\mathbb{N}}$ and

$$\{d_0\} <^\infty f_{s_0}, f_{s_0^c}.$$

In step $\alpha < \mathfrak{d}$. Note that $|\{d_\beta : \beta < \alpha\}| < \mathfrak{d}$, there exists a f_{s_α} such that $f_{s_\alpha^c} \in \mathbb{N}^{\mathbb{N}}$ and

$$\{d_\beta : \beta < \alpha\} <^\infty f_{s_\alpha}, f_{s_\alpha^c}.$$

The sketch of the proof

Proof.

It is not hard to see that $\{f_{s_\gamma} : \gamma < \mathfrak{d}\}$ is $\mathfrak{d}$-unbounded, so is the set $\{s_\gamma : \gamma < \mathfrak{d}\}$

Recall that the complement function τ: $\mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$

$$\tau(A) = \mathbb{N} \setminus A$$

is a homeomorphism. We define the following map ψ: $P(\mathbb{N}) \to \mathbb{N}^{\mathbb{N}}$ such that

$$\psi\colon\ s \to s^c \to f_{s^c}.$$

ψ is a continuous function, and $\psi(X \cup Fin)$ contains a unbounded family $\{f_{s^c_\gamma} : \gamma < \mathfrak{d}\}$.

Thus, $X \cup Fin$ is not a Hurewicz space. □

The sketch of the proof

Fact

$(X \cup Fin)^2$ is Menger.

Proof of the Fact

Proof.

Note that

$$(X \cup Fin)^2 = (X \times X) \cup (Fin \times Fin) \bigcup ((X \cup Fin) \times Fin) \bigcup (Fin \times (X \cup Fin)).$$

Define f: $P(\mathbb{N}) \to P(\mathbb{N}) \times P(\mathbb{N})$.
such that

$$f(x) = (x_0, x_1).$$

Where $x_0 = \{x(2n) : n < \mathbb{N}\}$, $x_1 = \{x(2n+1) : n < \mathbb{N}\}$. □

The sketch of the proof

Proof.

Then f is continuous and

$$f((X \bigoplus X) \cup Fin) = (X \times X) \cup (Fin \times Fin).$$

We need to show that

$$(X \bigoplus X) \cup Fin \text{ is } \mathfrak{d}\text{-concentrated on } Fin.$$

Notice that $X \bigoplus X$ homeomorphic to $\{f_{s_\alpha} \bigoplus f_{s_\beta} : \alpha, \beta < \mathfrak{d}\}$.

we just need to show that

$$\{f_{s_\alpha} \bigoplus f_{s_\beta} : \alpha, \beta < \mathfrak{d}\} \text{ is } \mathfrak{d}\text{-unbounded in } \mathbb{N}^{\mathbb{N}}.$$

Proof.

Let b be any element of $\mathbb{N}^{\mathbb{N}}$ and $b = b_0 \oplus b_1$.

where $b_0 = \{b(2n) : n \in \mathbb{N}\}$ and $b_1 = \{b(2n+1) : n \in \mathbb{N}\}$.
It is easy to see that

$$f_{s_\alpha} \oplus f_{s_\beta} \leq^* b \text{ if and only if } f_{s_\alpha} \leq^* b_0 \text{ and } f_{s_\beta} \leq^* b_1.$$

Then we have that
$\{f_{s_\alpha} \oplus f_{s_\beta} : f_{s_\alpha} \oplus f_{s_\beta} \leq^* b, \alpha, \beta < \mathfrak{d}\}$
$= \{f_{s_\alpha} : f_{s_\alpha} \leq^* b_0, \alpha < \mathfrak{d}\} \bigcap \{f_{s_\beta} : f_{s_\alpha} \leq^* b_1, \beta < \mathfrak{d}\}$.

□

The sketch of the proof

Proof.

Note that

$$|\{f_{s_\alpha} : f_{s_\alpha} \leq^* b_0, \alpha < \mathfrak{d}\}| < \mathfrak{d};$$
$$|\{f_{s_\beta} : f_{s_\alpha} \leq^* b_1, \beta < \mathfrak{d}\}| < \mathfrak{d}\ ;$$

Thus,

$$|\{f_{s_\alpha} \oplus f_{s_\beta} : f_{s_\alpha} \oplus f_{s_\beta} \leq^* b, \alpha, \beta < \mathfrak{d}\}| < \mathfrak{d}.$$

Then we have completed the proof of the **Fact** above. □

The sketch of the proof

Fact

For every $n > 2$, $(X \bigcup Fin)^n$ satisfies Menger property.

Proof.

For each n, Notice that $X \times Fin \times X$ is homeomorphic to $X \times X \times Fin$. We have that

$$(X \cup Fin)^k = \bigcup_{1 \leq k \leq n} C_n^k (X^k \times Fin^{n-k})$$

$$= \bigcup_{1 \leq k \leq n} C_n^k ((X^k \times Fin^{n-k}) \cup Fin^n)$$
$$= \bigcup_{1 \leq k \leq n} C_n^k ((X^k \cup Fin^k) \times Fin^{n-k})$$

It is enough to show that $X^k \cup Fin^k$ is Menger for each $1 \leq k \leq n$, and the proof is similar to that of the case $n = 2$ above.

□

The sketch of proof

Proof.

Define a function

$$P(\mathbb{N}) \to (P(\mathbb{N}))^k.$$

such that

$$f(x) = \langle x(0), x(1), \cdots x(n-1) \rangle.$$

Where $x(i) = \langle x(k) : k \equiv i \ mod(n) \rangle$ for each $1 \leq i \leq n$.

It is clearly that f is continuous.
Moreover.

$$f(\bigoplus_{1 \leq i \leq k} X_i \cup Fin) = X^k \cup Fin^k.$$

Where $X = X_i$ for each $i \leq k$.

The sketch of proof

Proof.

Observation: $\bigoplus_{1\leq i\leq k} X_i \cup Fin$ is $\mathfrak{d}$-concentrated on Fin.

To see this, recall that

$$\bigoplus_{1\leq i\leq k} X_i \approx \{\oplus_{j\in F} f_{s_j} : f_{s_j} \in X, F \in [\mathfrak{d}]^k\}.$$

Let b be any element of $\mathbb{N}^{\uparrow\mathbb{N}}$ and $b = b_0 \oplus b_1 \oplus \cdots \oplus b(n-1)$. Where

$$b_i = \{b(k) : k \equiv i\ mod(n)\}.$$

It is easy to see that

$$\bigoplus_{j\in F} f_{s_j} \leq^* b \text{ if and only if } f_{s_i} \leq^* b_i \text{ for each } i < n.$$

□

The sketch of the proof

Proof.

Then we have that

$$\{\oplus_{j\in F} f_{s_j} : \oplus_{j\in F} f_{s_j} \leq^* b\} = \bigcap_{i\leq n-1}\{f_{s_\alpha} : f_{s_\alpha} \leq^* b(i), \alpha < \mathfrak{d}\}.$$

Note that the space X is $\mathfrak{d}$-unbounded, so for all $i \leq n-1$, both the cardinalities of $\{f_{s_\alpha} : f_{s_\alpha} \leq^* b(i), \alpha < \mathfrak{d}\}$are less than $\mathfrak{d}$. Therefore,

$$|\{\oplus_{j\in F} f_{s_j} : \oplus_{j\in F} f_{s_j} \leq^* b,\ \oplus_{j\in F} f_{s_j} \in \bigoplus_{1\leq i\leq k} X_i\}| < \mathfrak{d}.$$

Thus, $\bigoplus_{1\leq i\leq k} X_i$ is $\mathfrak{d}$-unbounded, and then $\bigoplus_{1\leq i\leq k} X_i \bigcup Fin$ is $\mathfrak{d}$-concentrated on Fin as desired. □

Application

Definition

a $semifilter$ on an infinite countable X is a non-empty family $\mathcal{F} \subseteq [X]^{\mathbb{N}}$ containing all almost-supersets of its elements. A $filter$ is a $semifilter$ closed under finite intersections.

Definition

For a filter $\mathcal{F}$ on X, $\mathcal{B} \subseteq \mathcal{F}$ is called a $base$ of $\mathcal{F}$ if for each $F \in \mathcal{F}$, these is a $B \in \mathcal{B}$ with $B \subseteq F$. The character of a filter is the minimal cardinality of its bases.

Application

How are different among the property $S_{fin}(\mathcal{O},\mathcal{O})$, $U_{fin}(\mathcal{O},\Gamma)$, $U_{fin}(\mathcal{O},\Omega)$ and $S_{fin}(\Omega,\Omega)$ in the realm of filters?
Recall that

Proposition (D. Chodounsky, D. Repov$\check{s}$, L. Zdomskky, 2014, J.S.L)

Let $\mathcal{F}$ be a filter on $\mathbb{N}$. Then $\mathcal{F}$ is $Menger$ $(Hurewicz)$ then for all $0 < n < \mathbb{N}$, $\mathcal{F}^n$ is $Menger$ $(Hurewicz)$.

In the realm of filters,

$$U_{fin}(\mathcal{O},\Gamma) \longrightarrow U_{fin}(\mathcal{O},\Omega) \longleftrightarrow S_{fin}(\mathcal{O},\mathcal{O})$$

$$U_{fin}(\mathcal{O},\Omega) \longleftrightarrow S_{fin}(\Omega,\Omega)$$

Application

Lemma (R.Hernández-Gutiérrez, P. Szeptycki, 2015, Comm Math Universitatis Carolinae)

(1) $\mathfrak{b}$ *is the minimal character of a filter that is not Hurewicz.*

(2) $\mathfrak{d}$ *is the minimal character of a filter that is not Menger.*

Lemma (R.Hernández-Gutiérrez, P. Szeptycki, 2015, Comm Math Universitatis Carolinae)

($\mathfrak{b} = \mathfrak{d}$*) There exist a Menger filter of character* $\mathfrak{d}$ *that is not Hurewicz.*

Question (R.Hernández-Gutiérrez, P. Szeptycki, 2015, Comm Math Universitatis Carolinae)

Is there, in ZFC, a Menger filter of character $\mathfrak{d}$ that is not Hurewicz?

Application

Given a set $X \subset 2^{\mathbb{N}}$, let $\mathcal{I}_X$ be the ideal generated by finite union of branches $C_x = \{x|n : n \in \mathbb{N}\}$ where $x \in X$ and $\mathcal{F}_X$ be the dual filter.

Lemma (R. Hernández-Gutiérrez, P. Szeptycki, 2015, Comm Math Universitatis Carolinae)

Let $X \subset 2^{\mathbb{N}}$. Then

(1) *every finite power of X is Menger if and only if $\mathcal{F}_X$ is Menger.*

(2) *every finite power of X is Hurewicz if and only if $\mathcal{F}_X$ is Hurewicz.*

Application

Theorem (J. He, J. Yu, S. Zhang)

There exists, in ZFC, a Menger filter with character $\mathfrak{d}$ that is not Hurewicz.

Proof.

Let X be a finite power Menger space with size $\mathfrak{d}$ that is not Hurewicz(the existence was proved in the previous paragraph).

Note that the set $\{(C_x)^c : x \in X\}$ is the filter base for $\mathcal{F}_X$, and $|\{(C_x)^c : x \in X\}| = \mathfrak{d}$.

By the lemma above, the filter $\mathcal{F}_X$ meets the bill. □

Thank you!