Nonmeasurable images in Polish space with respect to $\sigma$-ideals with Borel base

Aleksander Cieślak and Robert Rałowski
Wrocław University of Science and Technology

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Let $X$ is a Polish space and $I \subseteq \mathcal{P}(X)$ s.t

- $I$ is $\sigma$-ideal with a Borel base and
- $I$ contains all singletons,

then $(X, I)$ is Polish ideal space.

Let $\mathcal{B}_+(I) = \text{Borel}(X) \setminus I$ be set of all $I$-positive Borel sets.

$\text{Perf}(X)$ stands for set of all perfect subsets of $X$. 
Notation and Terminology

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Definition (Cardinal coefficients)

Let $X$ - Polish space and $I \subseteq \mathcal{P}(X)$ be $\sigma$-ideal and $\mathcal{F} \subset I$ let

\[
cov(\mathcal{F}) = \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{F} \land \bigcup \mathcal{A} = X\}
\]

\[
cov_h(\mathcal{F}) = \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{F} \land (\exists B \in \mathcal{B}_+(I)) \bigcup \mathcal{A} = B\}
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cof(I) = \min\{|\mathcal{B}| : \mathcal{B} \subseteq I \land (\forall A \in I)(\exists B \in \mathcal{B}) A \subseteq B\}
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$\mathcal{N}$ $\sigma$-ideal of null sets and $\mathcal{M}$ $\sigma$-ideal of all meager subsets of $X$.

$\text{cov}(\mathcal{M}) = \text{cov}_h(\mathcal{M}), \text{cov}(\mathcal{N}) = \text{cov}_h(\mathcal{N})$.

Theorem (Cichoń-Kamburelis-Pawlikowski)

If $I$ is c.c.c. $\sigma$-ideal with Borel base then $\text{cof}(I) = \text{Cof}(I)$
Definition (Cardinal coefficients)

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$N$ $\sigma$-ideal of null sets and $M$ $\sigma$-ideal of all meager subsets of $X$. $cov(M) = cov_h(M)$, $cov(N) = cov_h(N)$.

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If $I$ is c.c.c. $\sigma$-ideal with Borel base then $\text{cof}(I) = \text{Cof}(I)$
Complete $I$-nonmeasurability

**Definition**
Let $(X, I)$ be Polish ideal space. We say that $A \subseteq X$ is completely $I$-nonmeasurable in $X$ iff

$$(\forall B \in B_+(I)) \ A \cap B \neq \emptyset \land A^c \cap B \neq \emptyset.$$ 

- $A \subseteq X$ is complete $[X]^{\leq \omega}$-nonmeasurable iff $A$ is Bernstein subset of $X$,
- $A \subseteq [0, 1]$ is complete $\mathcal{N}$-nonmeasurable iff $\lambda^*(A) = 0$ and $\lambda^*(B) = 1$,
- $A \subseteq X$ is complete $\mathcal{M}$-nonmeasurable if $\emptyset \neq U \subseteq X$ then $A \cap U$ does not have Baire property.
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Theorem
Let $X \subseteq X_0$ and \{ $Y_\alpha : \alpha < c$ \} be a Polish subspaces
Assume that $c$ is regular cardinal number.
If \{ $f_\alpha : \alpha < c$ \} be a family of functions such that for any $\alpha < c$
1. $f_\alpha[X] = Y_\alpha$,
2. for any $y \in \bigcup_{\alpha < c} Y_\alpha$ we have $f_\alpha^{-1}[y] \in [X]<c$.
Then there exists a subset $A \subseteq X$ such that for any $\alpha < c$ $f_\alpha[A]$ is a Bernstein set in $Y_\alpha$. 
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Then there exists a subset $A \subseteq X$ such that for any $\alpha < \mathfrak{c}$ $f_\alpha[A]$ is a Bernstein set in $Y_\alpha$. 
Corollary

There is subset $A \subset S^1$ of the unit circle that for any projection $\pi$ on real line $l \subseteq \mathbb{R}^2$ on the real plane of the set $A$ is a Bernstein set in $\pi[S^1]$.

Thus we have negative answer for

[asked Aug 3 ’11 at 7:51 simon 162] Suppose $A$ is contained in the unit square of $\mathbb{R}^2$, and the projection of $A$ on any line outside the unit square is not Lebesgue measurable in $\mathbb{R}$. Does that imply that $A$ is not Lebesgue measurable in the plane?

Moreover, our answer is valid for measure and category simultaneously.
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Example

Let

- $\mathcal{F} \subseteq P(\omega)$ - Frechet filter,
- $X = \omega^\omega$, $Y_C = \omega^C$ where $C \in \mathcal{F}$,
- $\omega^\omega \ni x \mapsto f_C(x) = x \upharpoonright C \in \omega^C$.

Then by the Theorem there is $A \subseteq \omega^\omega$ such that each image $f_C[A]$ is a Bernstein subset of $\omega^C$. 
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Remark

If we consider any function \( f : X \to X_0 \) such that \( f[X] \) is a Polish space, \( A \subseteq X \) Bernstein set then

1. if preimage of any singleton of \( f[X] \) contains a perfect set then \( f[A] = f[X] \),

2. if \( f \) is continuous then \( f[A] \) contains some Bernstein set in \( f[X] \) (because any preimage of perfect set in \( f[X] \) contains perfect set in \( X \)).
Remark

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1. $(\forall f \in \mathcal{F})(f[X] \subseteq X_0$ is Polish space),
2. $(\forall f \in \mathcal{F})(f : X \to X_0 \land I_f \subseteq P(f[X])$ be $\sigma$-ideal with Borel base on $f[X])$,
3. $|\mathcal{F}| \leq \sup\{\text{Cof}(I_f) : f \in \mathcal{F}\}$,
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Then there exists subset $A$ of $X$ such that for any $f \in \mathcal{F}$ the image $f[A]$ is completely $I_f$-nonmeasrable in $f[X]$. 
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Then there exists subset $A$ of $X$ such that for any $f \in \mathcal{F}$ the image $f[A]$ is completely $I_f$-nonmeasurable in $f[X]$. 
Theorem
Assume that \((X_0, I)\) is Polish ideal space and let \(X \subseteq X_0\) be \(I\)-positive Borel subset. Let \(\mathcal{F}\) be a family of functions such that

1. for every \(f \in \mathcal{F}\) the image \(f[X]\) is Borel subset of \(X_0\) and \(I_f \subseteq P([f[X]])\) is \(\sigma\)-ideal with Borel base in \(f[X]\),
2. \(|\mathcal{F}| \leq \max\{\text{Cof}(I), \sup\{\text{Cof}(I_f) : f \in \mathcal{F}\}\}\),
3. there is set \(Z \in I\) such that \(\text{Cof}(I) \leq \text{cov}\(\{f^{-1}[\{d]\}] : f \in \mathcal{F} \land d \in X_0 \setminus Z\}, I)\),
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Then there exists \(A \subseteq X\) which is completely \(I\)-nonmeasurable in \(X\) such that for every \(f \in \mathcal{F}\) the image \(f[A]\) is completely \(I_f\)-nonmeasurable in \(f[X]\).
Corollary

Assume MA. If $I \in \{\mathcal{N}, \mathcal{M}\}$ is a $\sigma$-ideal defined on Cantor space and $X \subset 2^\omega$ be a Borel $I$-positive. If $\mathcal{F}$ with at most size equal to $\mathfrak{c}$ and for any $f \in \mathcal{F}$ $\text{rng}(f)$ is Borel and $I_f \in \{\mathcal{N}, \mathcal{M}\}$ then the above two Theorems are true.
In the Mathoverflow webpage [2] the user Gowers gives positive answer for the following question

**[Gerald Edgar Aug 3 ’11 at 13:57]** (a) All projections but two are non-measurable? Or: (b) Projections in uncountably many directions measurable and projections in uncountably many other directions non-measurable?

The user of Mathoverflow asked:

**[answered Aug 3 ’11 at 14:47 gowers]** I don’t know what happens if we ask for continuum many measurable projections and continuum many non-measurable projections ...
**Theorem**

Let $c$ be regular, $X$, $Y$ be Polish spaces and

- $\{Y_\alpha : \alpha \in Y\}$ be a family of Polish spaces,
- $\{f_\alpha : \alpha \in Y\}$ be a family functions such that for all distinct $\alpha, \beta \in Y$
  - $\forall y \in Y_\alpha |f_\alpha^{-1}[y]| = c$
  - $\forall y \in Y_\alpha$ and $y' \in Y_\beta$ $|f_\alpha[y] \cap f_\beta[y']| < c$.

Then there exists a subset $A \subseteq X$ and disjoint Bernstein sets $F, G \subseteq Y$ such that $Y = F \cup G$ and

$$F = \{\alpha \in Y : f_\alpha[A] = Y_A\}$$

$$G = \{\alpha \in Y : f_\alpha[A] \text{ is Bernstein in } Y_\alpha\}.$$
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1. $\{Y_\alpha : \alpha \in Y\}$ be a family of Polish spaces,
2. $\{f_\alpha : \alpha \in Y\}$ be a family functions such that for all distinct $\alpha, \beta \in Y$
   1. $\forall y \in Y_\alpha \ |f_\alpha^{-1}[y]| = c$
   2. $\forall y \in Y_\alpha$ and $y' \in Y_\beta \ |f_\alpha[y] \cap f_\beta[y']| < c$.

Then there exists a subset $A \subseteq X$ and disjoint Bernstein sets $F, G \subseteq Y$ such that $Y = F \cup G$ and

$$F = \{\alpha \in Y : f_\alpha[A] = Y_A\}$$

$$G = \{\alpha \in Y : f_\alpha[A] \text{ is Bernstein in } Y_\alpha\}.$$
Fact

Let $n \geq 2$ be a fixed integer then every projection $\pi$ of the Lusin set $A \subseteq B(0, 1) \subseteq \mathbb{R}^n$ into tangent hyperplane $l$ to $B(0, 1)$ is Lusin set in $\pi[B(0, 1)]$. The same result is true if we replace Lusin set by Sierpiński set.
Fact

It is relatively consistent with ZFC theory that $\neg CH$ and for every integer $n \geq 2$ there exists Baire nonmeasurable subset $A$ of the cardinality less than $\mathfrak{c}$ of the unit ball $B \subseteq \mathbb{R}^n$ such that projection $\pi[A]$ into any tangent to $B$ hyperplane has not Baire property. The same result is true in the case of Lebesgue measure.
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**Theorem**

Let $X$ be a compact Polish space and $G \subseteq \mathcal{H}(X)$ be uncountable $G_\delta$ subset of $\mathcal{H}(X)$. Let $B \subseteq X$ be a comeager subset of $X$. Then there are perfect subsets $P \subseteq X$ and $Q \subseteq G$ such that for every homeomorphism $f \in Q$ of $X$ we have $P \subseteq f[B]$. 

Theorem
Let \( D \subseteq \mathbb{R}^2 \) be a unit disc with center in origin coordinates and \( B \subseteq D \) a comeager (or \( D \setminus B \) is null) set in \( D \). Then there are perfect set of directions \( R \) on \( \text{bd}(D) \) and \( P, Q \subseteq [-1, 1] \) such that

\[
(\forall \alpha \in R) \ (r_\alpha[P \times Q] \subseteq B),
\]

where \( r_\alpha \) is rotation by \( \alpha \) over origin of the real plane \( \mathbb{R}^2 \).

Theorem
Let \( n \geq 2 \) and \( B_n \subseteq \mathbb{R}^n \) be a \( n \)-dimensional unit ball. Let us assume that \( E \subseteq B \) a comeager (or \( B_n \setminus E \) is null) set in \( B_n \). Then there are perfect set \( R \) in \( D = \text{bd}(B_n) \), non-meager (non-null) \( P \subseteq B_{n-1} \) and \( Q \subseteq [-1, 1] \) such that

\[
(\forall \alpha \in R) \ (r_\alpha[P \times Q] \subseteq B_n),
\]

where \( r_\alpha \) is rotation of \( \alpha \) to the vector \((1, 0, \ldots, 0) \in \mathbb{R}^n \) over origin of the euclidean space \( \mathbb{R}^n \).
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Thank You

Mathoverflow: mathoverflow.net/questions/71976/lebesgue-non-measurability-in-the-plane
Thank You