# Luzin and Sierpiński sets meet trees

Marcin Michalski,
based on joint work with R. Rałowski & Sz. Żeberski

Wrocław University of Science and Technology

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## Definition

*Let* $T \subseteq \omega^{<\omega}$ *be a tree. Then*

- *for each* $\tau \in T$ $succ(\tau) = \{n \in \omega : \tau^\frown n \in T\}$;
- $split(T) = \{\tau \in T : |succ(\tau)| \geq 2\}$;
- $\omega$-$split(T) = \{\tau \in T : |succ(\tau)| = \aleph_0\}$.
- $stem(T) \in T$ *is a node* $\tau$ *such that for each* $\sigma \subsetneq \tau$ $|succ(\sigma)| = 1$ *and* $|succ(\tau)| > 1$.

## Definition

*A tree $T$ on $\omega$ is called*

- *a Sacks tree or perfect tree, denoted by $T \in \mathbb{S}$, if for each node $\sigma \in T$ there is $\tau \in T$ such that $\sigma \subseteq \tau$ and $|succ(\tau)| \geq 2$;*
- *a Miller tree or superperfect tree, denoted by $T \in \mathbb{M}$, if for each node $\sigma \in T$ exists $\tau \in T$ such that $\sigma \subseteq \tau$ and $|succ(\tau)| = \aleph_0$;*
- *a Laver tree, denoted by $T \in \mathbb{L}$, if for each node $\tau \supseteq stem(T)$ we have $|succ(\tau)| = \aleph_0$;*
- *a complete Laver tree, denoted by $T \in \mathbb{CL}$, if $T$ is Laver and $stem(T) = \emptyset$;*

**Definition (tree ideal $t_0$)**

*Let $\mathbb{T}$ be a family of trees. We say that a set $X$ belongs to the tree ideal $t_0$ if*

$$(\forall T \in \mathbb{T})(\exists T' \in \mathbb{T})(T' \subseteq T \ \& \ [T'] \cap X = \emptyset)$$

Let $h : \omega^\omega \to \mathbb{R} \setminus \mathbb{Q}$ be a homeomorphism between the Baire space and the space of irrational numbers.

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The classic example is Marczewski ideal $s_0$ for the family of perfect trees $\mathbb{S}$.
We will denote "Miller null" ideal by $m_0$, "Laver null" by $l_0$ and "complete Laver null" by $cl_0$.
For convenience purposes we will assume that bodies of trees already lie in $\mathbb{R}$.

Let $\mathcal{I}$ be an ideal in a Polish space $X$

**Definition**

*We call a set $L$ $\mathcal{I}$-Luzin set if $|L \cap A| < |L|$ for every set $A \in \mathcal{I}$.*

For classic ideals of Lebesgue null sets $\mathcal{N}$ and meager sets $\mathcal{M}$ we call $\mathcal{N}$-Luzin sets generalized Sierpiński sets and $\mathcal{M}$-Luzin sets generalized Luzin sets.

**Theorem (M., Rałowski, Żeberski 2017)**

*Let $\mathfrak{c}$ be a regular cardinal and let $t_0 \in \{s_0, m_0, l_0, cl_0\}$. Then for every generalized Luzin set $L$ and generalized Sierpiński set $S$ we have $L + S \in t_0$.*

**Lemma**

*There exists a dense $G_\delta$ set $G$ such that for every Miller (resp. Laver or complete Laver) tree $T$ there exists a Miller (resp. Laver or complete Laver) subtree $T'$ such that $G + [T'] \in \mathcal{N}$*

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so that $\tau_\sigma \subsetneq \tau_{\sigma^\frown k}$ for $\sigma \in (n+1)^{\leq n}$ and $\tau_\sigma \in \omega\text{-}split(T_n)$ for $\sigma \in (n+1)^{\leq n+1}$.

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- Analogously we do fusion in the case of Laver trees.

**Lemma**

*For every sequence of intervals $(I_n)_{n\in\omega}$ and a Miller (resp. Laver) tree $T$ there is a Miller (resp. Laver) fusion sequence $(T_n)_{n\in\omega}$ such that for all $n > 0$:*

$$\lambda([T_n] + I_n) < (1 + \Sigma_{k=0}^{n-1}(n-1)^k)\lambda(I_n).$$

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**Proof (idea of).**

By fusion and the fact that we always may find arbitrarily short interval which will cover infinitely many nodes (clopens generated on them) of a given split. □

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- $\mathbb{Q} = \{q_n : n \in \omega\}$ and let $I_n$'s be intervals with centers $q_n$'s with $\lambda(I_n) < \frac{1}{(n)^{n-1}2^n}$.

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- Hence $\lambda(\bigcup_{k>n} I_k + [T']) \leq \Sigma_{k>n}\lambda([T'] + I_k) \leq \Sigma_{k>n}\frac{1}{2^k} = \frac{1}{2^n}$.

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- So for $G = \bigcap_{n\in\omega}\bigcup_{k>n} I_k$ we have $\lambda(G + [T']) \leq lim_{n\to\infty}\frac{1}{2^n} = 0$.

□

**Theorem (Essentially Rothberger)**

*Assume that generalized Luzin set $L$ and generalized Sierpiński set $S$ exist. Then, if $\kappa = \max\{|L|, |S|\}$ is a regular cardinal, $|L| = |S| = \kappa$.*

**Theorem (M., Ratowski, Żeberski 2017)**

*Let $\mathfrak{c}$ be a regular cardinal and let $t_0 \in \{s_0, m_0, l_0, cl_0\}$. Then for every generalized Luzin set $L$ and generalized Sierpiński set $S$ we have $L + S \in t_0$.*

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- Let $L$ be a generalized Luzin set and $S$ generalized Sierpiński set. If $|L + S| < \mathfrak{c}$ then there is nothing to prove. Otherwise $|L| = |S| = \mathfrak{c}$ by regularity of $\mathfrak{c}$.

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- Let $t_0 = m_0$ and $T$ be a Miller tree. Let $T' \subseteq T$ and $G$ be as in the Lemma. Then for sets $A = -G$ and $B = ([T'] + G)^c$ we have $[T'] \subseteq (A + B)^c$

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- $L + S = (L \cap A) \cup (L \cap A^c) + (S \cap B) \cup (S \cap B^c)$.
- It follows that $|[T'] \cap L + S| < \mathfrak{c}$, so we may find a Miller tree $T'' \subseteq T'$ for which $T'' \cap (L + S) = \emptyset$.

□

**Thank you for your attention!**

M. Michalski, R. Rałowski, Sz. Żeberski, Nonmeasurable sets and unions with respect to tree ideals, arXiv:1712.05212