Cardinal characteristics and strong compactness

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Strong compactness vs Supercompactness

Definition

- $\kappa$ is **strongly compact** iff for all $\lambda \geq \kappa$ there is an elementary embedding $j : V \rightarrow M$ with $\text{crit}(j) = \kappa$ and there is $s \in M$ with $|s|^M < j(\kappa)$ and $j'' \lambda \subseteq s$.

- $\kappa$ is **supercompact** iff for all $\lambda \geq \kappa$ there is an elementary embedding $j : V \rightarrow M$ with $\text{crit}(j) = \kappa$ and $\lambda^M \subseteq M$. 
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Supercompactness $\iff$ Strong compactness
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\[
\text{Supercompactness} \quad \implies \quad \text{Strong compactness}
\]

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\begin{align*}
\text{Con}(\text{ZFC+there is a supercompact}) & \quad \implies \quad \text{Con}(\text{ZFC+there is a strongly compact}) \\
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Is it consistent/possible to control the cardinal characteristics of a non-supercompact strongly compact cardinal?
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Question

Is it consistent/possible to control the cardinal characteristics of a non-supercompact strongly compact cardinal?

The result in this talk is a stepping stone to a positive answer.
If $\kappa$ is a regular uncountable cardinal and $\kappa^+ < 2\kappa$, a **cardinal characteristic** refers to a combinatorial property, such that the least size of a subset of $\mathcal{P}(\kappa)$ or $\kappa^\kappa$ that satisfies it is between $\kappa^+$ and $2\kappa$.
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- There is ongoing research on generalising the known cardinal characteristics of the continuum.
- It is connected to the study of the generalised Baire space $\kappa^{\kappa}$ or $2^\kappa$.
- Sometimes we need large cardinals in order to control the value of cardinal characteristics.
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- Every regular uncountable cardinal $\kappa$ carries uniform ultrafilters (extend the dual filter of $[\kappa]^{<\kappa}$ to an ultrafilter using Zorn’s lemma).
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- Every regular uncountable cardinal $\kappa$ carries uniform ultrafilters (extend the dual filter of $[\kappa]^{<\kappa}$ to an ultrafilter using Zorn’s lemma).
- $\kappa^+ \leq u(\kappa) \leq 2^\kappa$.
- It is unclear how to control the base of an arbitrary uniform ultrafilter with forcing.
- That is why we prefer to work with **measurable** cardinals.
Theorem

A cardinal \( \kappa \) is measurable iff it is the critical point of an elementary embedding \( j : V \rightarrow M \).
Measurable cardinals

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- For the $\leftarrow \rightarrow$ direction, if $j : V \rightarrow M$ has critical point $\kappa$, then we can define an ultrafilter $U$ on $\kappa$ by

  $$X \in U \iff \kappa \in j(X).$$

- It is easy to see that $U$ is uniform.
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- Since there is a plethora of results about lifting embeddings in forcing extensions, it seems more promising to control the ultrafilter number of a measurable cardinal.
Theorem (Brooke-Taylor, Fischer, Friedman, Montoya - 2017)

Suppose $\kappa < \kappa^* \leq \lambda$ are regular cardinals and $\kappa$ is supercompact. Then, there is a forcing extension inside which $\kappa$ remains supercompact, $u(\kappa) = \kappa^*$ and $2^\kappa = \lambda$. 

Supercompactness is a much stronger property than measurability. The proof relies on the indestructibility of supercompact cardinals. It is worth asking if the large cardinal assumption can be reduced. The forcing notion used is an iteration of Mathias forcing.
Suppose $\kappa < \kappa^* \leq \lambda$ are regular cardinals and $\kappa$ is supercompact. Then, there is a forcing extension inside which $\kappa$ remains supercompact, $\mu(\kappa) = \kappa^*$ and $2^\kappa = \lambda$.

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- The forcing notion used is an iteration of Mathias forcing.
Iterated Mathias forcing

For a regular $\kappa > \omega$ and $U$ an ultrafilter on $\kappa$,

$$\mathbb{M}^\kappa_U = \{ \langle s, A \rangle \mid s \in \kappa^{<\kappa}, A \in U \}$$

$$\langle t, B \rangle \leq \langle s, A \rangle \text{ iff } t \supseteq s, B \subseteq A \text{ and } t - s \in A.$$
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We do an iteration $\mathbb{M}$ where at each stage an ultrafilter $U$ is chosen in order to force with $\mathbb{M}_U^\kappa$. We allow the generic filter to decide which ultrafilter to use, i.e. we force with the lottery sum

$$\bigoplus_{U \text{ ultrafilter on } \kappa} \mathbb{M}_U^\kappa$$
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Key Lemma (BT-F-F-M)

Subject to technical assumptions on the iteration, if $U$ is an ultrafilter on $\kappa$ in $\mathcal{V}^\mathcal{M}$, then it has been chosen quite often in the lottery.
As we mentioned, the previous theorem relied heavily on the indestructibility of supercompactness.

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Indestructible strong compactness

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**Theorem (Apter & Gitik - 1998)**

Assume the existence of a supercompact cardinal, it is consistent that the first strongly compact cardinal $\kappa$ is also the first measurable cardinal and indestructible under $\kappa$-directed closed forcing.
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*Assume the existence of a supercompact cardinal, it is consistent that the first strongly compact cardinal $\kappa$ is also the first measurable cardinal and indestructible under $\kappa$-directed closed forcing.*

- Note that under GCH, the first measurable does not possess any degree of supercompactness!
- We aim to adapt the proof of the previous theorem in the Apter-Gitik model.
- Note that this does not improve the consistency strength, but is rather an indication of a possible improvement.
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**Proof sketch:**

1. Fix a Laver function $f: \kappa \rightarrow V_\kappa$.
2. Perform the usual Laver preparation $P$ with respect to $f$ to make $\kappa$ indestructible under $\kappa$-directed closed forcing with the following addition.
3. As the iteration proceeds, we also perform Prikry forcing to destroy all measurable cardinals below $\kappa$.
4. In the final model, $\kappa$ has no measurable cardinals below it. With Prikry-type arguments we can show it remains strongly compact.
5. With Laver-style arguments we can show that $\kappa$ becomes indestructible under $\kappa$-directed closed forcing.
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Theorem (D., 2018)

Suppose \( \kappa \) is a supercompact cardinal and \( \kappa < \kappa^* \leq \lambda \) are regular cardinals with \( \lambda^\kappa = \lambda \) and \( \lambda^\kappa = \lambda \). Then, there is a forcing extension in which \( \kappa \) is the first strongly compact and the first measurable, \( \mathfrak{u}(\kappa) = \kappa^* \) and \( 2^\kappa = \lambda \).
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Proof sketch:

- Start with the Apter-Gitik variation of Laver preparation $\mathbb{P}$ that makes $\kappa$ the first strongly compact and indestructible (iterated Prikry forcing is involved).
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- As $\kappa$ is indestructible, we can further force with the Mathias iteration $\mathbb{M}$ of BT-F-F-M.
- For a large enough $\lambda$, choose a $\lambda$-supercompactness embedding $j : V \rightarrow M$ with $j(f)(\kappa) = \mathbb{M}$. 
By elementarity,
\[ j(P \ast M) \simeq P \ast M \ast (\text{Prikry iteration}) \ast (\text{tail of } j(P)) \ast j(M). \]
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Force the generic filters for the Prikry iteration and the tail forcing to first lift \( j \) through \( \mathbb{P} \).
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Using Prikry-type arguments, show that \( U \) is already definable in \( V^{\mathbb{P} \ast \mathbb{M}} \).
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Combining the Key Lemma of BT-F-F-M with a small tease of the master condition, we can form a base of the desired size for \( U \).

The previous step shows that \( u(\kappa) \leq \kappa^* \). For the converse, we use the fact that the Mathias generics form an unbounded family.
Question 1

What about other cardinal characteristics?

In the model of BT-F-F-M, all other cardinal characteristics in Cichoń's diagram are also equal to $u_p \kappa_q$.

Corollary

In the forcing extension constructed before, $\text{add} p_M \kappa_q \text{cof} p_M \kappa_q \text{non} p_M \kappa_q \text{cov} p_M \kappa_q \text{b} p_\kappa q \text{d} p_\kappa q u_p \kappa_q$.

Not known (to my knowledge) if the cardinals in Cichoń's diagram can be controlled independently to the ultrafilter number.
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Question 2

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Question 2

Can we prove a similar result for an arbitrary strongly compact cardinal?

- It is still open whether we can violate GCH at a strongly compact, without assuming supercompactness ($\kappa^{++}$-supercompactness is the best known consistency bound).
- Some specific cases may be easier to handle (e.g. a measurable limit of supercompact cardinals).
Question 3

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Question 3

*Can we reduce the assumption of supercompactness in the BT-F-F-M theorem?*

- We need more than measurability (to violate GCH at a measurable cardinal we need at least a Mitchell rank 2 measurable.)
- Strong cardinals would be a good candidate: we can violate GCH and there is a certain degree of indestructibility.
Thank you for listening!