Grothendieck $C(K)$-spaces of small density

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Weak topologies

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Grothendieck spaces

Inclusions of the weak topologies on X^*

$$\textit{weak}^* \quad \subseteq \quad \textit{weak} \quad \subseteq \quad \textit{norm}$$

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Definition

An infinite dimensional Banach space X is *Grothendieck* if every weak* convergent sequence in the dual X^* is weakly convergent.

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Examples of Grothendieck spaces

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- $C(K)$ such that $C(K) = c_0 \oplus Y$ for some closed subspace Y

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- algebra $\mathcal{J}$ of Jordan-measurable subsets of $[0, 1]$

The Grothendieck number

The Grothendieck number $\mathfrak{gr}$

$\mathfrak{gr} = \min\left\{|\mathcal{A}|\colon\ \text{infinite } \mathcal{A} \text{ has the Grothendieck property}\right\}$

$$\omega_1 \leqslant \mathfrak{gr} \leqslant \mathfrak{c}$$

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Problem

Describe $\mathfrak{gr}$ in terms of classical cardinal characteristics of the continuum.

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Recall that: $|\mathcal{A}| = w(St(\mathcal{A}))$.

Corollary

If $|\mathcal{A}| < \max(\mathfrak{s}, \mathrm{cov}(\mathcal{M}))$, then $\mathcal{A}$ does not have the Grothendieck property. Hence, $\mathfrak{gr} \geqslant \max(\mathfrak{s}, \mathrm{cov}(\mathcal{M}))$.

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Corollary

If CH holds in V and G is a $\mathbb{S}^\kappa$-generic filter over V, then $\mathfrak{gr} = \omega_1 < \kappa = \mathfrak{c}$ holds in $V[G]$.

Generalization

Definition

A forcing $\mathbb{P} \in V$ has *the Laver property* if for every $\mathbb{P}$-generic filter G over V, every $f \in \omega^\omega \cap V$ and $g \in \omega^\omega \cap V[G]$ such that $g \leqslant^* f$, there exists $H\colon \omega \to [\omega]^{<\omega}$ such that $g(n) \in H(n)$ and $|H(n)| \leqslant n + 1$ for every $n \in \omega$.

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Recall that: Con($\mathfrak{r} = \mathfrak{u} < \mathfrak{s}$) and Con($\mathfrak{g} < \mathrm{cov}(\mathcal{M})$)

Corollary

No ZFC inequality between $\mathfrak{gr}$ and any of the numbers $\mathfrak{r}$, $\mathfrak{u}$ and $\mathfrak{g}$.

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Question

$\mathrm{Con}(\mathfrak{d} < \mathfrak{gr})$?

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If $\mathrm{cof}([\mathrm{cof}(\mathcal{N})]^\omega) = \mathrm{cof}(\mathcal{N})$, then $\mathfrak{gr} \leqslant \mathrm{cof}(\mathcal{N})$.

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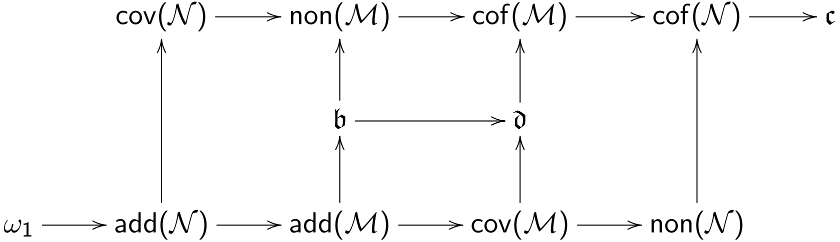
Recall that $\mathrm{Con}(\omega_2 = \mathrm{cof}(\mathcal{N}) < \mathfrak{a} = \omega_3)$ (Brendle '03).

Corollary

No ZFC inequality between $\mathfrak{gr}$ and $\mathfrak{a}$.

Cichoń's diagram and $\mathfrak{gr}$



What's known:

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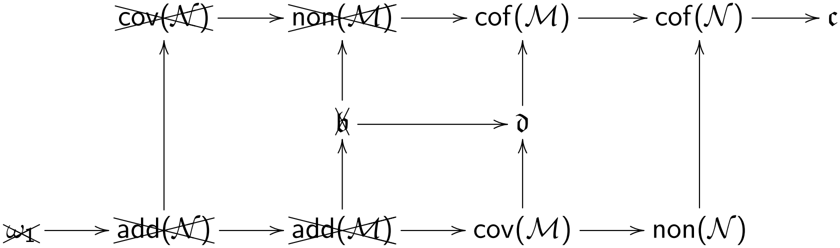

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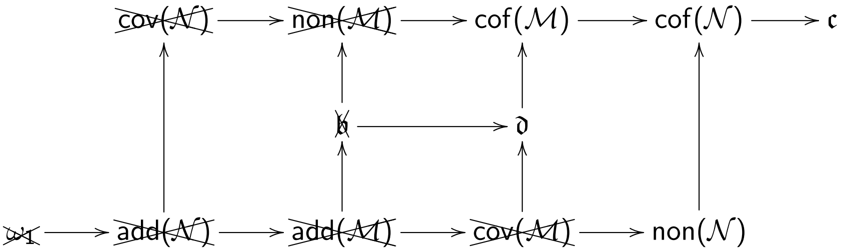

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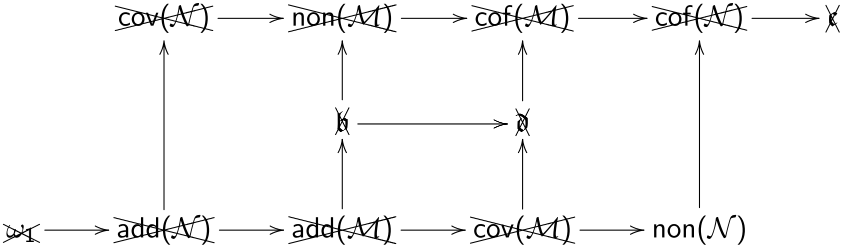

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3. $\mathrm{Con}(\mathfrak{gr} < \mathfrak{d})$

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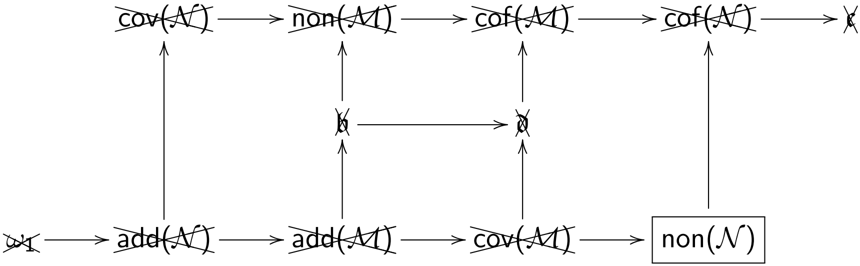

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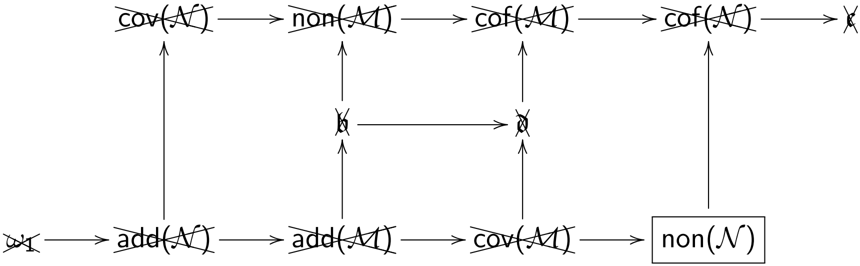

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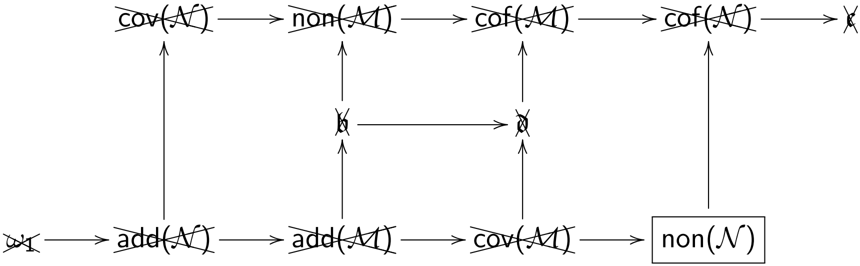

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3. Con($\mathfrak{gr} <$ cov($\mathcal{N}$))? (the random model?)

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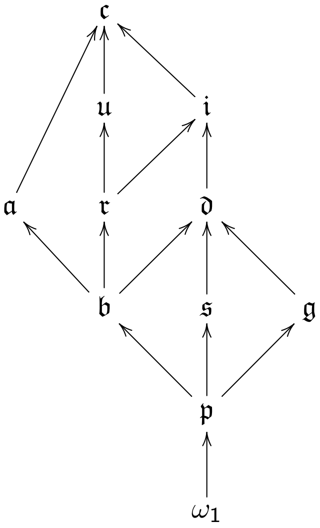

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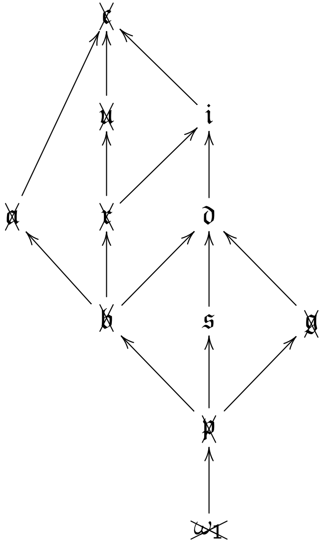

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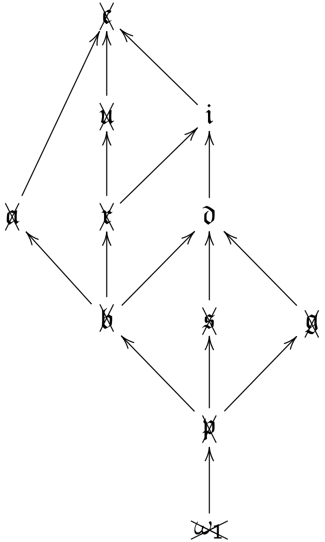

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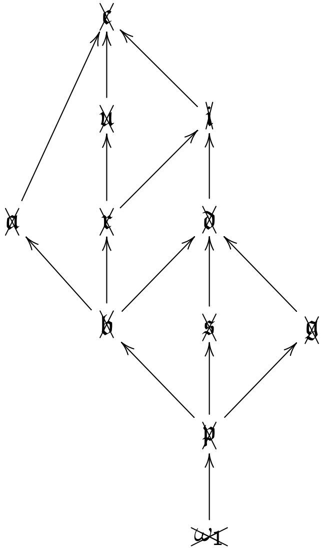

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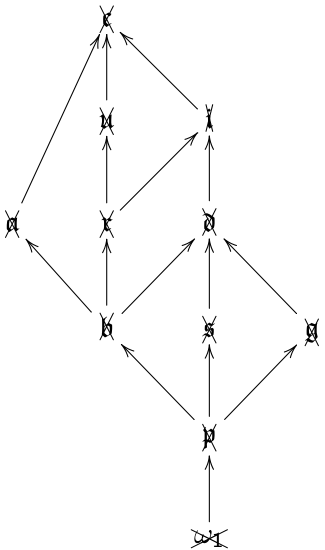
3. $\mathrm{Con}(\mathfrak{gr} < \mathfrak{d})$

Van Douwen's diagram and $\mathfrak{gr}$



What's known:

1. No ZFC ineq. between $\mathfrak{gr}$ and $\mathfrak{r}$, $\mathfrak{u}$, $\mathfrak{g}$ and $\mathfrak{a}$
2. $\mathrm{cov}(\mathcal{M}) \leqslant \mathfrak{gr}$ and $\mathrm{Con}(\mathfrak{s} < \mathrm{cov}(\mathcal{M}))$
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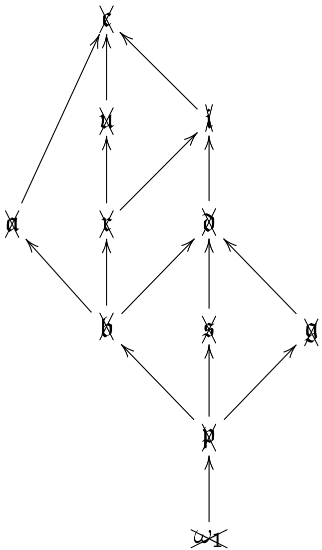
Van Douwen's diagram and $\mathfrak{gr}$



What's known:

1. No ZFC ineq. between $\mathfrak{gr}$ and $\mathfrak{r}$, $\mathfrak{u}$, $\mathfrak{g}$ and $\mathfrak{a}$
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3. $\mathrm{Con}(\mathfrak{gr} < \mathfrak{d})$
4. $\mathfrak{s} \leqslant \mathfrak{gr}$

Questions:

1. $\mathfrak{b} \leqslant \mathfrak{gr}$?
2. $\mathfrak{gr} \leqslant \mathfrak{d}$?

A word on the cofinality of $\mathfrak{gr}$

Theorem (Schachermayer '82)

$\mathrm{cf}(\mathfrak{gr}) > \omega$.

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Fact

$\mathfrak{gr}$ may be either regular (CH) or singular (in every model where $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} > \mathrm{cf}(\mathfrak{c})$).

The end

Thank you for the attention!