Separation axiom for regular closed sets

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Background

O. Frink gave the following characterization of completely regular spaces (1964), compare Engelking's book Exercise 1.5.G.

Theorem (O. Frink)

A T_1-space X is completely regular iff there exists a base $\mathcal{B}$ satisfying the following condition:

- *For every $x \in X$ and every $U \in \mathcal{B}$ that contains x there exists a $V \in \mathcal{B}$ such that $x \notin V$ and $U \cup V = X$.*
- *For any $U, V \in \mathcal{B}$ satisfying $U \cup V = X$, there exist $U^*, V^* \in \mathcal{B}$ such that $X \setminus V \subset U^*$ and $X \setminus U \subset V^*$ and $U^* \cap V^* = \emptyset$.*

If a space is completely regular, then the base consisting of all co-zero sets satisfies Frink's characterization. The rest of the proof repeats a proof of Urysohn's lemma.

Normal spaces, regular open sets

Obviously, if X is normal, then the family of all open sets fulfils both conditions in Frink's characterization, *i.e.*, one can consider the topology as a base $\mathcal{B}$.

Mathematicians working in the field of Boolean algebras and their Stone spaces consider regular open sets, in fact, bases consisting of all regular open sets. A number of papers examine a space with a base (of closed subsets) consisting of regular closed sets, which satisfies Frink's conditions.

Question

When does the family of all regular open sets satisfy Frink's conditions?

It appears to us that there is a gap in the literature, since we could not find any information concerning non-normal counterexamples like the Niemytzki plane, the Sorgenfrey plane, the Tychonoff plank *etc*.

One-point extensions

P. Kalemba and Sz. Plewik (arXiv:1701.04322) have examined methods by which a regular but not completely regular space can be obtained, using one-point extensions (of a completely regular space). Then, at the seminar in Katowice, the following question was asked:

Question

Does there exist a regular but not completely regular one-point extension of the Niemytzki plane?

It appears that there is no such extension, *i.e.*, every regular one-point extension of the Niemytzki plane is completely regular, since the family of all regular open sets satisfies Frink's conditions.

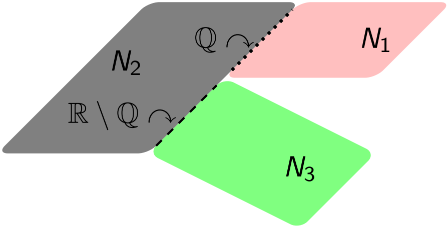
Announced axiom

We say that a completely regular space is an *rc-space*, if every two disjoint regular closed subsets have disjoint open neighbourhoods.

Fact

Every regular one-point extension of an rc-space is completely regular. □

Assume that N_i is a copy of the Niemytzki plane. Then subspaces N_1 and N_3 are regular closed, but they have no disjoint open neighbourhoods.



Examples of rc-spaces

Example

The following examples are rc-spaces:

- the Niemytzki plane,
- the Tychonoff plank

$$([0, \omega_1] \times [0, \omega]) \setminus \{(\omega_1, \omega)\},$$

- the Sorgenfrey plane.

Example

The property of being an rc-space is not hereditary: any Hausdorff compactification of any completely regular space is normal, hence an rc-space.

The Niemytzki plane

In 2007 D. Chodounský characterized all pairs of closed subsets of the Niemytzki plane which can be separated by open neighbourhoods.

Theorem (D. Chodounský, 2007)

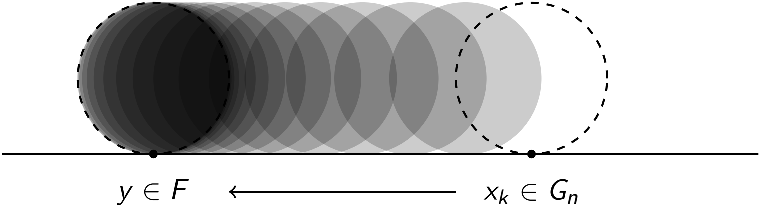
Let $L = \{(x,0)\colon x \in \mathbb{R}\}$ *be a subset of the Niemytzki plane* N. *Disjoint closed subsets* $F, G \subseteq N$ *can be separated if and only if there exist families* $\{F_n\colon n < \omega\}$ *and* $\{G_n\colon n < \omega\}$ *such that* $F \cap L = \bigcup_{n<\omega} F_n$, $G \cap L = \bigcup_{n<\omega} G_n$ *and*

$$F \cap \mathrm{cl}_{\mathbb{R}}\, G_n = \mathrm{cl}_{\mathbb{R}}\, F_n \cap G = \emptyset.$$

By $\mathrm{cl}_{\mathbb{R}}$ we denote the closure with respect to the natural topology of the real line. One can prove that if subsets F, G of the Niemytzki plane are regular closed, then F, G have disjoint open neighbourhoods.

The Niemytzki plane is an rc-space

Let N be the Niemytzki plane and $L = \{(x, 0) \colon x \in \mathbb{R}\}$.
Fix disjoint regular closed subsets $F, G \subseteq N$. For every $x \in G \cap L$ there is a radius $r_x > 0$ such that $B(x + r_x, r_x) \cap F = \emptyset$. For every n, let $G_n = \{x \in G \cap L \colon r_x \geqslant \frac{1}{n}\}$. Then $G \cap L = \bigcup_n G_n$ and for every n, $F \cap \mathrm{cl}_{\mathbb{R}} G_n = \emptyset$. Similarly, we can define F_n and prove that $\mathrm{cl}_{\mathbb{R}} F_n \cap G = \emptyset$.



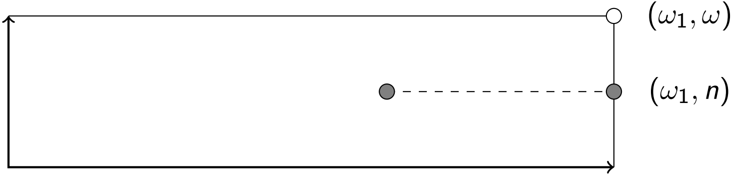
For every $(x_k)_k \subseteq G_n$ converging to y, we have $B(y + \frac{1}{n}, \frac{1}{n}) = \bigcup_k B(x_k + \frac{1}{n}, \frac{1}{n})$.

The Tychonoff plank is an rc-space

The subspace $T = ([0, \omega_1] \times [0, \omega]) \setminus \{(\omega_1, \omega)\}$ of the product $P = [0, \omega_1] \times [0, \omega]$ is called *the Tychonoff plank*.
Fix two open subsets $U, V \subseteq T$ such that $(\omega_1, \omega) \in \mathrm{cl}_P\, U \cap \mathrm{cl}_P\, V$.
It suffices to show that the sets

$$\{\alpha < \omega_1 \colon (\alpha, \omega) \in \mathrm{cl}_P\, U\}, \quad \{\alpha < \omega_1 \colon (\alpha, \omega) \in \mathrm{cl}_P\, V\}$$

are club in $[0, \omega_1)$, since it implies that there exists $(\alpha, \omega) \in \mathrm{cl}_P\, U \cap \mathrm{cl}_P\, V$.
Fix $\alpha < \omega_1$. For every $n < \omega$ there exists $(\beta_n, m_n) \in \mathrm{cl}_P\, U$ such that $\alpha < \beta_n < \omega_1$ and $n < m_n$. We can assume that $\beta_n < \beta_{n+1}$.
Then $\beta = \sup_n \beta_n < \omega_1$ and $(\beta, \omega) \in \mathrm{cl}_P\, U$.



In an rc-space disjoint regular closed subsets can be separated

Fix disjoint regular closed subsets F, G of an rc-space X.
There exist disjoint open sets U, V such that $F \subseteq U$ and $G \subseteq V$.
Then $\operatorname{cl} U$ and G are disjoint and regular closed, hence there exist disjoint open subsets U', V' such that $\operatorname{cl} U \subseteq U'$ and $G \subseteq V'$.
Thus we have

$$F \subseteq U \subseteq \operatorname{cl} U \subseteq X \setminus G.$$

We can assume that U is regular open (take $\operatorname{int} \operatorname{cl} U$ instead of U).
We continue using the standard procedure from the proof of Urysohn's lemma.

Every regular one-point extension of an rc-space is completely regular

Fix an rc-space X, its regular extension $Y = X \cup \{t\}$ and a closed subset $F \subseteq Y$.
If $t \notin F$, then there exist disjoint open subsets $U, V \subseteq Y$ such that $F \subseteq U$ and $t \in V$.
We can assume that $\operatorname{cl} U \cap \operatorname{cl} V = \emptyset$.
We obtain the desired continuous function using the fact that in an rc-space disjoint regular closed subsets can be separated.
Fix $x \in X \setminus F$.
Let W be an open subset such that $t \in W$ and $x \notin \operatorname{cl} W$.
Then there exists a continuous $f \colon X \to \mathbb{R}$ such that $f(x) = 1$ and $f \restriction ((F \cup \operatorname{cl} W) \cap X) = 0$. We extend f assuming $f(t) = 0$.

Fact

Every one-point regular extension of a normal space is normal.

Fix a normal space X and a regular space $Y = X \cup \{p\}$. Fix disjoint closed subsets $F, G \subseteq X \cup \{p\}$. We can assume that $p \notin F$. Thus F is a closed subset of X. Since $X \cup \{p\}$ is regular, there exists a neighbourhood U of p such that $F \cap \operatorname{cl}_Y U = \emptyset$. Subsets F and $G \cap X$ are closed in X and disjoint, hence there exist disjoint open subsets $V, W \subseteq X$ such that $F \subseteq V$ and $G \cap X \subseteq W$. Then $V = V' \cap X$ and $W = W' \cap X$ for some V', W' open in Y. Observe that $F \subseteq V \setminus \operatorname{cl}_Y U = V' \setminus \operatorname{cl}_Y U$, $G \subseteq W \cup U = W' \cup U$ and $V' \setminus \operatorname{cl}_Y U, W' \cup U$ are disjoint and open in Y.

One-point extensions of completely regular spaces

Fact

Assume that $X \cup \{p\}$ is a one-point extension of a completely regular space X such that $X \cap \operatorname{cl} U$ is compact for some neighbourhood U of p. Then $X \cup \{p\}$ is also completely regular.

A. Zame, *A note on Wallman spaces*. Proc. Amer. Math. Soc. 22 (1969) 141–144.

D. Chodounský, *Non-normality and relative normality of Niemytzki plane*, Acta Universitatis Carolinae. Mathematica et Physica, Vol. 48 (2007), No. 2, 37–41.

P. Kalemba, Sz. Plewik, *On regular but not completely regular spaces*, `arXiv:1701.04322`.

Thank you for your attention.