Separation axiom for regular closed sets

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O. Frink gave the following characterization of completely regular spaces (1964), compare Engelking’s book Exercise 1.5.G.

**Theorem (O. Frink)**

A $T_1$-space $X$ is completely regular iff there exists a base $B$ satisfying the following condition:

- For every $x \in X$ and every $U \in B$ that contains $x$ there exists a $V \in B$ such that $x \notin V$ and $U \cup V = X$.

- For any $U, V \in B$ satisfying $U \cup V = X$, there exist $U^*, V^* \in B$ such that $X \setminus V \subset U^*$ and $X \setminus U \subset V^*$ and $U^* \cap V^* = \emptyset$.

If a space is completely regular, then the base consisting of all co-zero sets satisfies Frink’s characterization. The rest of the proof repeats a proof of Urysohn’s lemma.
Obviously, if $X$ is normal, then the family of all open sets fulfils both conditions in Frink’s characterization, i.e., one can consider the topology as a base $\mathcal{B}$.

Mathematicians working in the field of Boolean algebras and their Stone spaces consider regular open sets, in fact, bases consisting of all regular open sets. A number of papers examine a space with a base (of closed subsets) consisting of regular closed sets, which satisfies Frink’s conditions.

**Question**

When does the family of all regular open sets satisfy Frink’s conditions?

It appears to us that there is a gap in the literature, since we could not find any information concerning non-normal counterexamples like the Niemytzki plane, the Sorgenfrey plane, the Tychonoff plank etc.
P. Kalemba and Sz. Plewik (arXiv:1701.04322) have examined methods by which a regular but not completely regular space can be obtained, using one-point extensions (of a completely regular space). Then, at the seminar in Katowice, the following question was asked:

**Question**

Does there exist a regular but not completely regular one-point extension of the Niemytzki plane?

It appears that there is no such extension, i.e., every regular one-point extension of the Niemytzki plane is completely regular, since the family of all regular open sets satisfies Frink’s conditions.
We say that a completely regular space is an \textit{rc-space}, if every two disjoint regular closed subsets have disjoint open neighbourhoods.

\textbf{Fact}

\textit{Every regular one-point extension of an rc-space is completely regular.}

Assume that $N_i$ is a copy of the Niemytzki plane. Then subspaces $N_1$ and $N_3$ are regular closed, but they have no disjoint open neighbourhoods.
Examples of rc-spaces

Example

The following examples are rc-spaces:

- the Niemytzki plane,
- the Tychonoff plank

\(((0, \omega_1] \times [0, \omega)) \setminus \{(\omega_1, \omega)\},\)

- the Sorgenfrey plane.

Example

The property of being an rc-space is not hereditary: any Hausdorff compactification of any completely regular space is normal, hence an rc-space.
In 2007 D. Chodounský characterized all pairs of closed subsets of the Niemytzki plane which can be separated by open neighbourhoods.

**Theorem (D. Chodounský, 2007)**

Let \( L = \{(x, 0) : x \in \mathbb{R}\} \) be a subset of the Niemytzki plane \( N \). Disjoint closed subsets \( F, G \subseteq N \) can be separated if and only if there exist families \( \{F_n : n < \omega\} \) and \( \{G_n : n < \omega\} \) such that

\[
F \cap L = \bigcup_{n<\omega} F_n, \quad G \cap L = \bigcup_{n<\omega} G_n \quad \text{and} \quad F \cap \overline{R} G_n = \overline{R} F_n \cap G = \emptyset.
\]

By \( \overline{R} \) we denote the closure with respect to the natural topology of the real line. One can prove that if subsets \( F, G \) of the Niemytzki plane are regular closed, then \( F, G \) have disjoint open neighbourhoods.
Let $N$ be the Niemytzki plane and $L = \{(x,0): x \in \mathbb{R}\}$. Fix disjoint regular closed subsets $F, G \subseteq N$. For every $x \in G \cap L$ there is a radius $r_x > 0$ such that $B(x + r_x, r_x) \cap F = \emptyset$. For every $n$, let $G_n = \{x \in G \cap L: r_x \geq \frac{1}{n}\}$. Then $G \cap L = \bigcup_n G_n$ and for every $n$, $F \cap \text{cl}_\mathbb{R} G_n = \emptyset$. Similarly, we can define $F_n$ and prove that $\text{cl}_\mathbb{R} F_n \cap G = \emptyset$.

For every $(x_k)_k \subseteq G_n$ converging to $y$, we have $B(y + \frac{1}{n}, \frac{1}{n}) = \bigcup_k B(x_k + \frac{1}{n}, \frac{1}{n})$. 

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The Tychonoff plank is an rc-space

The subspace \( T = ([0, \omega_1] \times [0, \omega]) \setminus \{(\omega_1, \omega)\} \) of the product \( P = [0, \omega_1] \times [0, \omega] \) is called the Tychonoff plank.

Fix two open subsets \( U, V \subseteq T \) such that \((\omega_1, \omega) \in \text{cl}_P U \cap \text{cl}_P V\).

It suffices to show that the sets

\[
\{ \alpha < \omega_1 : (\alpha, \omega) \in \text{cl}_P U \}, \quad \{ \alpha < \omega_1 : (\alpha, \omega) \in \text{cl}_P V \}
\]

are club in \([0, \omega_1)\), since it implies that there exists \((\alpha, \omega) \in \text{cl}_P U \cap \text{cl}_P V\).

Fix \( \alpha < \omega_1 \). For every \( n < \omega \) there exists \((\beta_n, m_n) \in \text{cl}_P U\) such that \( \alpha < \beta_n < \omega_1 \) and \( n < m_n \). We can assume that \( \beta_n < \beta_{n+1} \).

Then \( \beta = \sup_n \beta_n < \omega_1 \) and \((\beta, \omega) \in \text{cl}_P U\).
In an rc-space disjoint regular closed subsets can be separated.

Fix disjoint regular closed subsets $F, G$ of an rc-space $X$. There exist disjoint open sets $U, V$ such that $F \subseteq U$ and $G \subseteq V$. Then $\text{cl} \, U$ and $G$ are disjoint and regular closed, hence there exist disjoint open subsets $U', V'$ such that $\text{cl} \, U \subseteq U'$ and $G \subseteq V'$. Thus we have

$$F \subseteq U \subseteq \text{cl} \, U \subseteq X \setminus G.$$ 

We can assume that $U$ is regular open (take $\text{int} \, \text{cl} \, U$ instead of $U$). We continue using the standard procedure from the proof of Urysohn’s lemma.
Every regular one-point extension of an rc-space is completely regular

Fix an rc-space $X$, its regular extension $Y = X \cup \{t\}$ and a closed subset $F \subseteq Y$.
If $t \notin F$, then there exist disjoint open subsets $U, V \subseteq Y$ such that $F \subseteq U$ and $t \in V$.
We can assume that $\text{cl} U \cap \text{cl} V = \emptyset$.
We obtain the desired continuous function using the fact that in an rc-space disjoint regular closed subsets can be separated.
Fix $x \in X \setminus F$.
Let $W$ be an open subset such that $t \in W$ and $x \notin \text{cl} W$.
Then there exists a continuous $f: X \rightarrow \mathbb{R}$ such that $f(x) = 1$ and $f \upharpoonright ((F \cup \text{cl} W) \cap X) = 0$. We extend $f$ assuming $f(t) = 0$. 
Fact

Every one-point regular extension of a normal space is normal.

Fix a normal space $X$ and a regular space $Y = X \cup \{p\}$. Fix disjoint closed subsets $F, G \subseteq X \cup \{p\}$. We can assume that $p \notin F$. Thus $F$ is a closed subset of $X$. Since $X \cup \{p\}$ is regular, there exists a neighbourhood $U$ of $p$ such that $F \cap \text{cl}_Y U = \emptyset$.

Subsets $F$ and $G \cap X$ are closed in $X$ and disjoint, hence there exist disjoint open subsets $V, W \subseteq X$ such that $F \subseteq V$ and $G \cap X \subseteq W$. Then $V = V' \cap X$ and $W = W' \cap X$ for some $V', W'$ open in $Y$. Observe that $F \subseteq V \setminus \text{cl}_Y U = V' \setminus \text{cl}_Y U$, $G \subseteq W \cup U = W' \cup U$ and $V' \setminus \text{cl}_Y U, W' \cup U$ are disjoint and open in $Y$. 
Fact

Assume that $X \cup \{p\}$ is a one-point extension of a completely regular space $X$ such that $X \cap \text{cl } U$ is compact for some neighbourhood $U$ of $p$. Then $X \cup \{p\}$ is also completely regular.


Thank you for your attention.