Products of CW complexes

Andrew Brooke-Taylor

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Bringing set theory and algebraic topology together
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So, focus on **CW complexes**: spaces built up by gluing on Euclidean discs of higher and higher dimension.

For \( n \in \mathbb{N} \), let

- \( D^n \) denote the closed ball of radius 1 about the origin in \( \mathbb{R}^n \) (the \( n \)-disc),
- \( \overset{\circ}{D^n} \) its interior (the open ball of radius 1 about the origin), and
- \( S^{n-1} \) its boundary (the \( n - 1 \)-sphere).
Definition

A Hausdorff space $X$ is a **CW complex** if there exists a set of continuous functions $\varphi^\alpha_n : D^n \to X$ (characteristic maps), for $\alpha$ in an arbitrary index set and $n \in \mathbb{N}$ a function of $\alpha$, such that:

1. $\varphi^\alpha_n \upharpoonright D^n$ is a homeomorphism to its image, and $X$ is the disjoint union as $\alpha$ varies of these homeomorphic images $\varphi^\alpha_n[D^n]$ (“cells”).

Closure-finiteness:

For each $\varphi^\alpha_n$, $\varphi^\alpha_n[S^n - S^{n-1}]$ is contained in finitely many cells all of dimension less than $n$.

Weak topology:

A set is closed if and only if its intersection with each closed cell $\varphi^\alpha_n[D^n]$ is closed.

We often denote $\varphi^\alpha_n[D^n]$ by $e^\alpha_n$. 
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We often denote $\varphi^n_\alpha [D^n]$ by $e^n_\alpha$. 

Let $X$ be the “star” with a central vertex $x_0$ and countably many edges $e_1, e_2, \ldots$ emanating from it (and the countably many “other end” vertices of those edges).

$X$ is not metrizable, as $x_0$ does not have a countable neighbourhood base.

Proof

Identify each edge with the unit interval, with $x_0$ at 0. Then for every $f: \mathbb{N} \to \mathbb{N}$, consider the open neighbourhood $U(x_0; f)$ of $x_0$ whose intersection with $e_1, e_2, \ldots$ is the interval $[0, 1/(f(n) + 1)]$.

These form a neighbourhood base, but for any countably many $f_i$, there is a $g$ that eventually dominates each of them, so $U(x_0; g)$ does not contain any of the $U(x_0; f_i)$.
Let $X$ be the “star” with a central vertex $x_0$ and countably many edges $e_{X,n}^1$ ($n \in \mathbb{N}$) emanating from it (and the countably many “other end” vertices of those edges).

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\[ \square \]
The Cartesian product of two CW complexes $X$ and $Y$, with the product topology, need not be a CW complex. Since $D^m \times D^n \cong D^{m+n}$, there is a natural cell structure on $X \times Y$, which satisfies closure-finiteness, but the product topology is generally not as fine as the weak topology.

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In this talk, $X \times Y$ is always taken to have the product topology, so "$X \times Y$ is a CW complex" means "the product topology on $X \times Y$ is the same as the weak topology".
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Consider the subset of $X \times Y$:

$$H = \{ (1^n + 1, 1^n + 1) \in e_{X,n}^1 \times e_{Y,f}^1 : n \in \mathbb{N}, f \in \mathbb{N}^\mathbb{N} \}$$

where we have identified each edge with the unit interval, with 0 at the centre vertex.

Since every cell of $X \times Y$ contains at most one point of $H$, $H$ is closed in the weak topology.
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Let $X$ be the “star” with a central vertex $x_0$ and countably many edges $e^{1}_{X,n} (n \in \mathbb{N})$ emanating from it (and the countably many “other end” vertices of those edges).

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Then \( \left( \frac{1}{g(k)+1}, \frac{1}{g(k)+1} \right) \in U \times V \cap H \). So in the product topology, \((x_0, y_0) \in \bar{H} \).
A *subcomplex* $A$ of a CW complex $X$ is what you would expect.
More preliminaries: subcomplexes

A subcomplex $A$ of a CW complex $X$ is a subspace which is a union of cells of $X$, such that if $e^n_\alpha \subseteq A$ then its closure $\bar{e}^n_\alpha = \varphi^n_\alpha[D^n]$ is contained in $A$. 

E.g. For any CW complex $X$ and $n \in \mathbb{N}$, the $n$-skeleton $X^n$ of $X$ is the subcomplex of $X$ which is the union of all cells of $X$ of dimension at most $n$. Every subcomplex $A$ of $X$ is closed in $X$. By closure-finiteness, every $x$ in a CW complex $X$ lies in a finite subcomplex.

Definition Let $\kappa$ be a cardinal. We say that a CW complex $X$ is locally less than $\kappa$ if for all $x$ in $X$ there is a subcomplex $A$ of $X$ with fewer than $\kappa$ many cells such that $x$ is in the interior of $A$. We write locally finite for locally less than $\aleph_0$, and locally countable for locally less than $\aleph_1$. 

Andrew Brooke-Taylor (Leeds)
A *subcomplex* $A$ of a CW complex $X$ is a subspace which is a union of cells of $X$, such that if $e^n_\alpha \subseteq A$ then its closure $\overline{e^n_\alpha} = \varphi_\alpha^n[D^n]$ is contained in $A$.

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**Definition**

Let \( \kappa \) be a cardinal. We say that a CW complex \( X \) is locally less than \( \kappa \) if for all \( x \) in \( X \) there is a subcomplex \( A \) of \( X \) with fewer than \( \kappa \) many cells such that \( x \) is in the interior of \( A \). We write locally finite for locally less than \( \aleph_0 \), and locally countable for locally less than \( \aleph_1 \).
Proposition

If $\kappa$ is a regular uncountable cardinal, then a CW complex $W$ is locally less than $\kappa$ if and only if every connected component of $W$ has fewer than $\kappa$ many cells.

Proof sketch.

$\Leftarrow$ is trivial. For $\Rightarrow$, given any point $w$, recursively fill out to get an open (hence clopen) subcomplex containing $w$ with fewer than $\kappa$ many cells, using the fact that the cells are compact to control the number of cells along the way. \hfill $\square$
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**Theorem (J. Milnor, 1956)**

*If $X$ and $Y$ are both (locally) countable, then $X \times Y$ is a CW complex.*
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If $X$ and $Y$ are both (locally) countable, then $X \times Y$ is a CW complex.

**Theorem (Y. Tanaka, 1982)**

If neither $X$ nor $Y$ is locally countable, then $X \times Y$ is not a CW complex.
What was known, beyond ZFC

Theorem (Liu Y.-M., 1978)

Assuming CH, $X \times Y$ is a CW complex if and only if either

- one of them is locally finite, or
- both are locally countable.

Theorem (Y. Tanaka, 1982)

Assuming $b = \aleph_1$, $X \times Y$ is a CW complex if and only if either

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Question

Can we show, without assuming any extra set-theoretic axioms, that the product $X \times Y$ of CW complexes $X$ and $Y$ is a CW complex if and only if either

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Answer (follows from Tanaka’s work)
No.
Can we nevertheless do better?

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Can we characterise exactly when the product of two CW complexes is a CW complex, without assuming any extra set-theoretic axioms?
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Answer (B.-T.)

Yes!
In the argument for Dowker’s example, there was a lot of inefficiency — we can do better, with the bigger star $Y$ potentially having fewer edges.
Pushing Dowker’s example harder

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**Recall:**

For $f, g \in \mathbb{N}^\mathbb{N}$, write $f \leq^* g$ if for all but finitely many $n \in \mathbb{N}$, $f(n) \leq g(n)$. I’ll write $f \leq g$ to mean that for all $n$, $f(n) \leq g(n)$.

The **bounding number** $b$ is the least cardinality of a set of functions that is unbounded with respect to $\leq^*$, i.e. such that no one $g$ is $\geq^*$ them all, i.e.,

$$b = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathbb{N}^\mathbb{N} \land \forall g \in \mathbb{N}^\mathbb{N} \exists f \in \mathcal{F} \neg (f \leq^* g)\}.$$  

$\aleph_1 \leq b \leq 2^{\aleph_0}$, and each inequality can be strict.
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Consider the subset of $X \times Y$

$$H = \left\{ \left( \frac{1}{f(n) + 1}, \frac{1}{f(n) + 1} \right) \in e_{X,n}^1 \times e_{Y,f}^1 : n \in \mathbb{N}, f \in \mathbb{N}^\mathbb{N} \right\}$$

where we have identified each edge with the unit interval, with 0 at the centre vertex.

Since every cell of $X \times Y$ contains at most one point of $H$, $H$ is closed in the weak topology.
Example (Folklore based on Dowker, 1952)

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Let $Y$ be the “star” with a central vertex $y_0$ and $b$ many edges $e_{Y,f}^1$ ($f \in \mathcal{F}$) emanating from it where $\mathcal{F} \subseteq \mathbb{N}^\mathbb{N}$ is unbounded w.r.t. $\leq^*$ (and the other ends).

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Then \( \left( \frac{1}{g(k)+1}, \frac{1}{g(k)+1} \right) \in U \times V \cap H \). So in the product topology, \((x_0, y_0) \in \bar{H} \).
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Let \( U \times V \) be a member of the open neighbourhood base about \((x_0, y_0)\) in the product topology on \( X \times Y \) — so \( x_0 \in U \) an open subset of \( X \), and \( y_0 \in V \) an open subset of \( Y \).

Consider the edges \( e_{X,n}^1 \) of \( X \):

Let \( g : \mathbb{N} \to \mathbb{N}^+ \) be an increasing function such that \([0, 1/g(n)) \subset e_{X,n}^1 \cap U\) for every \( n \in \mathbb{N} \). Take \( f \in \mathcal{F} \) such that \( f \nleq^* g \).

Consider the edge \( e_{Y,f}^1 \) of \( Y \):

Let \( k \in \mathbb{N} \) be such that \( \frac{1}{f(k) + 1} \in e_{Y,f}^1 \cap V \) and \( f(k) > g(k) \).

Then \( \left( \frac{1}{f(k) + 1}, \frac{1}{f(k) + 1} \right) \in U \times V \cap H \). So in the product topology, \((x_0, y_0) \in \bar{H} \).
Is this harder-working Dowker example optimal?
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Yes!
Theorem (B.-T.)

Let $X$ and $Y$ be CW complexes. Then $X \times Y$ is a CW complex if and only if one of the following holds:

1. $X$ or $Y$ is locally finite.
2. One of $X$ and $Y$ is locally countable, and the other is locally less than $\mathfrak{b}$. 
Proof

follows from the work of Tanaka (1982).

locally finite case: Whitehead (1949).

So it remains to show that if $X$ and $Y$ are CW complexes such that $X$ is locally countable and $Y$ is locally less than $b$, then $X \times Y$ is a CW complex.

By the Proposition earlier, we may assume that $X$ has countably many cells and $Y$ has fewer than $b$ many cells.
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By the Proposition earlier, we may assume that $X$ has countably many cells and $Y$ has fewer than $b$ many cells.
Any compact subset of a CW complex $X$ is contained in finitely many cells, and each closed cell $\bar{e}_n^\alpha$ is compact. So

$$X \text{ has the weak topology } \iff \text{ the topology is } compactly \ generated$$

i.e. a set is closed if and only if its intersection with every compact set is closed.
Topologies

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We can also restrict to those compact sets which are continuous images of the space $\omega + 1$ (with the order topology).

Definition

A topological space $Z$ is **sequential** if for every subset $C$ of $Z$, $C$ is closed if and only if $C$ contains the limit of every convergent (countable) sequence from $C$ — $C$ is **sequentially closed**.
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**Definition**

A topological space $Z$ is *sequential* if for every subset $C$ of $Z$, $C$ is closed if and only if $C$ contains the limit of every convergent (countable) sequence from $C$ — $C$ is *sequentially closed*.

Any sequential space is compactly generated. Since $D^n$ is sequential for every $n$, we have that CW complexes are sequential.
Need to show: $X \times Y$ is sequential.
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So suppose

- $H \subset X \times Y$ is sequentially closed, and
- $(x_0, y_0) \in X \times Y \setminus H$.

We want to construct open neighbourhoods $U$ of $x_0$ in $X$ and $V$ of $y_0$ in $Y$ such that $(U \times V) \cap H = \emptyset$. 
Constructing neighbourhoods

We can build an open neighbourhood $U$ of a point $x$ in a CW complex $X$ by induction on dimension:
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We can build an open neighbourhood $U$ of a point $x$ in a CW complex $X$ by induction on dimension:

- If $x \in e^n_\alpha \subset X$, start with the image under $\varphi^n_\alpha$ of an open ball in $\overset{\circ}{D}^n$. 

Lemma: Such open neighbourhoods form a base for the topology on $X$. 

Wrinkle in proof. Use compactness of closed cells.
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- Once $U \cap X^k$ is defined, for each $(k + 1)$-cell $e_{\beta}^{k+1}$ whose boundary intersects $U \cap X^k$, take a collar neighbourhood of $(\varphi_{\beta}^{k+1})^{-1}(U \cap X^k) \subseteq S^k = \partial D^{k+1}$.
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For any function $f$ from the set of indices of cells in $X$ to $\mathbb{N}$ we thus get an open neighbourhood $U(x; f)$, taking radius/width $\frac{1}{f(\beta)+1}$ for the cell $\beta$ step.
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**Lemma**

*Such open neighbourhoods form a base for the topology on $X$.*

**Wrinkle in proof.**

Use compactness of closed cells.
Constructing neighbourhoods avoiding \( H \)

Lemma 1 (Adding one cell to finite subcomplexes)

Suppose \( W \) and \( Z \) are CW complexes, \( W' \) is a finite subcomplex of \( W \), \( Z' \) is a finite subcomplex of \( Z \), \( U \subseteq W' \) is open in \( W' \), \( V \subseteq Z' \) is open in \( Z' \), and \( H \) is a sequentially closed subset of \( W \times Z \) such that the closure of \( U \times V \) is disjoint from \( H \).

Let \( e \) be a cell of \( Z \) whose boundary is contained in \( Z' \). Then there is a \( p \in \mathbb{N} \) such that, if \( V^e, p \) is \( V \) extended by the width \( 1/(p+1) \) collar in \( e \), then \( U \times V^e, p \) has closure disjoint from \( H \).

Proof sketch.

Use compactness, normality and sequentiality of \( W' \times (Z' \cup e) \).
Constructing neighbourhoods avoiding $H$

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- $H$ is a sequentially closed subset of $W \times Z$ such that the closure of $U \times V$ is disjoint from $H$.

Let $e$ be a cell of $Z$ whose boundary is contained in $Z'$. Then there is a $p \in \mathbb{N}$ such that, if $V^{e,p}$ is $V$ extended by the width $1/(p + 1)$ collar in $e$, then $U \times V^{e,p}$ has closure disjoint from $H$. 

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Andrew Brooke-Taylor (Leeds)
Constructing neighbourhoods avoiding $H$

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Back to the proof of the Theorem

We want to construct open neighbourhoods \( U \) of \( x_0 \) in \( X \) and \( V \) of \( y_0 \) in \( Y \) such that \( (U \times V) \cap H = \emptyset \).

Basic idea
Simultaneous induction on cell number on the \( X \) side (after enumerating the cells of \( X \) in a reasonable order) and dimension on the \( Y \) side. For each new cell \( e_\alpha \) that you consider on the \( Y \) side, you get a function \( f_\alpha : N \to N \) defining an open set on the \( X \) side avoiding \( H \). Since there are fewer than \( b \) many \( \alpha \), they can be eventually dominated by a single function \( f \), with respect to which the \( e_\alpha \) part of the neighbourhood can be chosen.

This doesn't work (\( f_\alpha \leq \ast f \) isn't good enough).
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We shall construct functions $f : \mathbb{N} \to \mathbb{N}$ and $g : J \to \mathbb{N}$, where $J$ is the index set for cells of $Y$, such that $U(x_0; f) \times U(y_0; g)$ has closure disjoint from $H$. 
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This doesn’t work ($f_\alpha \leq^* f$ isn’t good enough).
\( \leq^* \) isn’t good enough

If \( f_\alpha(n) \leq f(n) \) for all \( n \), then \( U(x; f_\alpha) \supseteq U(x; f) \).

For 1-dimensional examples (Dowker, Tanaka), this isn’t a big deal. For arbitrary CW complexes, where higher dimensional cells can glue on to those finitely many cells, it’s a problem.

Solution

Hechler conditions!

Andrew Brooke-Taylor (Leeds)  Products of CW complexes
$\leq^*$ isn't good enough

If $f_\alpha(n) \leq f(n)$ for all $n$, then $U(x; f_\alpha) \supseteq U(x; f)$.

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**Solution**

Hechler conditions!
The construction is actually by recursion on dimension on the $Y$ side, and simultaneously, constructing $f$ as the limit of a descending sequence of Hechler conditions, that is:

- finite initial segments of $f$, and
- promises to dominate some function $F$ thereafter.
Lemma 2 (Adding a $Y$-side cell, fitting $X$-side promises)

Let $Y'$ be a finite subcomplex of $Y$ containing $y_0$, $F: \mathbb{N} \to \mathbb{N}$ be a function, $i \in \mathbb{N}$, and $s$ be a function from the indices of $Y'$ to $\mathbb{N}$ such that $U(x_0; F) \times U(y_0; s) \subseteq X \times Y'$ has closure disjoint from $H$, $Y'' = Y' \cup e_\alpha$ for some cell $e_\alpha$ of $Y$ not in $Y'$.

Then there is a function $f: \mathbb{N} \to \mathbb{N}$ such that

1. $f(n) \geq F(n)$ for all $n$ in $\mathbb{N}$,

2. for every $f': \mathbb{N} \to \mathbb{N}$ such that $f' \geq \ast f$ and $f' \geq F$, there is a $q \in \mathbb{N}$ such that $U(x_0; f') \times U(y_0; s \cup \{(\alpha, q)\})$ has closure disjoint from $H$. 

Andrew Brooke-Taylor (Leeds)
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Then there is a function $f : \mathbb{N} \to \mathbb{N}$ such that

1. $f(n) \geq F(n)$ for all $n$ in $\mathbb{N}$, and $f(n) = F(N)$ for all $n < i$,

2. for every $f' : \mathbb{N} \to \mathbb{N}$ such that $f' \geq^* f$ and $f' \geq F$, there is a $q \in \mathbb{N}$ such that $U(x_0; f') \times U(y_0; s \cup \{ (\alpha, q) \})$ has closure disjoint from $H$. 
Proof of Lemma 2

For every finite tuple $r$ of length $n$ such that $r \geq F \upharpoonright n$, $U(x_0; r) \subset U(x_0; F)$, so $U(x_0; r) \times U(y_0; s)$ certainly has closure disjoint from $H$. 
Proof of Lemma 2

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By Lemma 1, we can then take $q_r \in \mathbb{N}$ such that $U(x_0; r) \times U(y_0; s \cup \{(\alpha, q_r)\})$ has closure disjoint from $H$. 
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By Lemma 1, we can then take $q_r \in \mathbb{N}$ such that $U(x_0; r) \times U(y_0; s \cup \{(\alpha, q_r)\})$ has closure disjoint from $H$.

Then by Lemma 1 again, there is $p \in \mathbb{N}$ sucht that $U(x_0; r \cup \{(n, p)\}) \times U(y_0; s \cup \{(\alpha, q_r)\})$ has closure disjoint from $H$. 
Now, assuming by induction we have defined $f \upharpoonright n$ ($n \geq i$), there are only finitely many $r$ with $F \upharpoonright n \leq r \leq f \upharpoonright n$; follow this procedure for all of them, and take the maximum of the resulting values $p$ to be $f(n)$.

Then for any $f' \geq F$ with $f' \geq^* f$, $f' \geq r \cup (f \upharpoonright [n, \infty))$ for some $n \geq i$ and some $r$ of length $n$ as above, so

$$U(x_0; f' \upharpoonright n + 1) \times U(y_0; s \cup \{(\alpha, q_r)\})$$ has closure disjoint from $H$,

and in fact

$$U(x_0; f') \times U(y_0; s \cup \{(\alpha, q_r)\})$$ has closure disjoint from $H$.

Lemma 2
Finishing the proof of the Theorem

With Lemma 2 in hand, the argument now follows as outlined before:

Proceed by induction of dimension on the $Y$ side. Assume we have defined $f_k : \mathbb{N} \to \mathbb{N}$ and $g \upharpoonright Y^k$. For each $(k + 1)$-dimensional cell $e_\alpha$ on the $Y$ side, use Lemma 2 with $f_k$ as $F$, $k$ as $i$, the minimal (finite) subcomplex of $Y$ containing $e_\alpha$ as $Y''$, and $g \upharpoonright (Y'' \setminus e_\alpha)$ as $s$ to get $f_{a_l,k+1}$. There are fewer than $b$ many such $f_{\alpha,k+1}$, so take $f_{k+1}$ eventually dominating all of them. Then take $q$ as given by Lemma 2 (with $f_{k+1}$ as $f'$) as $g(\alpha)$.

Finally, take $f$ to be the (componentwise) limit of the $f_{k+1}$; these $f$ and $g$ are such that $U(x_0; f) \times U(y_0; g)$ has closure disjoint from $H$. 

\[\square\]