# Compactifiability and Borel complexity up to equivalence

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# Our questions

We say that classes of topological spaces $\mathcal{C}$, $\mathcal{D}$ are *equivalent* if every member of $\mathcal{C}$ is homeomorphic to a member of $\mathcal{D}$ and vice versa. We write $\mathcal{C} \cong \mathcal{D}$.

Let $\mathcal{C}$ be a class of metrizable compacta.

**Question**

Can $\mathcal{C}$ be disjointly composed into one metrizable compactum such that the quotient space is also a metrizable compactum?

If $\mathcal{C}$ is a class of continua, then the question is equivalent to the following.

**Question**

Is there a metrizable compactum such that its set of connected components is equivalent to $\mathcal{C}$?

# Compactifiable and Polishable classes

### Definition

A class of topological spaces $\mathcal{C}$ is called

- *compactifiable* if there is a continuous map $q\colon A \to B$ between metrizable compacta such that $\{q^{-1}(b) : b \in B\} \cong \mathcal{C}$,
- *Polishable* if there is a continuous map $q\colon A \to B$ between Polish spaces such that $\{q^{-1}(b) : b \in B\} \cong \mathcal{C}$.

# Compositions – the witnessing objects

More generally, a *composition* $\mathcal{A}$ consists of the following data:

$$\{A_b\}_{b\in B} \xrightarrow{\{e_b\}_{b\in B}} A \xrightarrow{\ q\ } B$$

- a *composition space* $A$ and an *indexing space* $B$,
- a family $\{A_b\}_{b\in B}$ of topological spaces being composed,
- a family of embeddings $\{e_b \colon A_b \hookrightarrow A\}_{b\in B}$ such that $\{\mathrm{rng}(e_b)\}_{b\in B}$ is a decomposition of $A$,
- a *composition map* $q \colon A \to B$ that is continuous and satisfies $q^{-1}(b) = \mathrm{rng}(e_b)$ for every $b \in B$.

We write $\mathcal{A}(q \colon A \to B)$ or $\mathcal{A} = (A, e_b)_{b\in B}$ or even $(A, A_b)_{b\in B}$ when $A_b \subseteq A$. For the latter cases, the induced composition map is denoted by $q_{\mathcal{A}}$.

# Compositions – the witnessing objects

A composition $\mathcal{A}(q\colon A \to B)$ is called

- *compact* if $A$, $B$ are metrizable compacta,
- *Polish* if $A$, $B$ are Polish spaces.

Let $A$, $B$ be topological spaces, let $F \subseteq A \times B$.

- We put $F^b := \{a \in A : (a, b) \in F\}$ for every $b \in B$.
- $F$ induces the composition $\mathcal{A}_F(\pi_B\colon F \to B)$.
- On the other hand, we may move from a composition $\mathcal{A}(q\colon A \to B)$ to the graph $\{(a, q(a)) : a \in A\} \subseteq A \times B$.

# Equivalences

## Theorem

The following conditions are equivalent for a class of spaces $\mathcal{C}$.

1. $\mathcal{C}$ is compactifiable.
2. There are metrizable compacta $A$, $B$ and a closed set $F \subseteq A \times B$ such that $\{F^b \colon b \in B\} \cong \mathcal{C}$.
3. There is a closed set $F \subseteq [0,1]^\omega \times 2^\omega$ such that $\{F^b : b \in 2^\omega\} \cong \mathcal{C}$.

1. $\mathcal{C}$ is Polishable.
2. There is a Polish space $A$, an analytic space $B$, and a $G_\delta$ set $F \subseteq A \times B$ such that $\{F^b \colon b \in B\} \cong \mathcal{C}$.
3. There is a $G_\delta$ set $F \subseteq [0,1]^\omega \times \omega^\omega$ such that $\{F^b : b \in \omega^\omega\} \cong \mathcal{C}$.
4. There is a closed set $F \subseteq (0,1)^\omega \times \omega^\omega$ such that $\{F^b : b \in \omega^\omega\} \cong \mathcal{C}$.

# Observations

- Compactifiable and Polishable classes are stable under countable unions – consider the one-point compactification of

$$\sum_{i\in I} q_i \colon \sum_{i\in I} A_i \to \sum_{i\in I} B_i.$$

- Hence, every countable family of metrizable compacta (or Polish spaces) is compactifiable (or Polishable).
- On the other hand, a cardinal argument gives that there are many classes of metrizable compacta that are not Polishable.
  - There are $\mathfrak{c}$-many $G_\delta$ subsets of $[0,1]^\omega \times \omega^\omega$.
  - There are $\mathfrak{c}$-many non-homeomorphic metrizable compacta, and so $2^\mathfrak{c}$-many non-equivalent classes.

# Hyperspaces

For a topological space $X$ we shall consider the hyperspaces of all subsets $\mathcal{P}(X)$, all closed subsets $\mathcal{C}l(X)$, all compact subsets $\mathcal{K}(X)$, and all subcontinua $\mathcal{C}(X)$ endowed with the Vietoris topology.

## Recall

- The Vietoris topology is generated by the sets
  $U^- = \{A \subseteq X : A \cap U \neq \emptyset\}$ and $U^+ = \{A \subseteq X : A \subseteq U\}$
  for open $U \subseteq X$.
- $\mathcal{K}(X)$ is metrizable by the Hausdorff metric.
- $\mathcal{K}(X)$ is compact (or Polish) if $X$ is compact (or Polish).
- $\mathcal{C}(X)$ is closed in $\mathcal{K}(X)$.
- $\mathcal{R}_\in = \{(x, A) : x \in A \in \mathcal{C}l(X)\}$ is closed in $X \times \mathcal{C}l(X)$.

# Compositions and hyperspaces

## Definition

A composition $\mathcal{A}(q\colon A \to B)$ is *strong* if $q$ is closed and open and $|B \setminus \operatorname{rng}(q)| \leq 1$. We also define *strongly compactifiable* and *strongly Polishable* classes.

## Construction

$$X,\ \mathcal{F} \subseteq \mathcal{P}(X) \xrightarrow{\ \mathcal{A}_{\mathcal{F}}(\pi_{\mathcal{F}}\colon \mathcal{R}_{\in} \cap (X \times \mathcal{F}) \to \mathcal{F})\ } \mathcal{A}(q\colon A \to B)$$

$$\mathcal{A}(q\colon A \to B) \xrightarrow{\ A,\ \mathcal{F}_{\mathcal{A}} := \{q^{-1}(b)\}_{b \in B} \subseteq \mathcal{P}(A)\ } X,\ \mathcal{F} \subseteq \mathcal{P}(X)$$

- If $\mathcal{F} \subseteq \mathcal{K}(X)$, then $\mathcal{A}_{\mathcal{F}}$ is strong.
- A composition $\mathcal{A}$ is strong if and only if $\mathcal{F}_{\mathcal{A}} \cong B$ via $q^{-1*}$.

# Compositions and hyperspaces

**Theorem**

The following conditions are equivalent for a class of spaces $\mathcal{C}$.

1. $\mathcal{C}$ is strongly compactifiable.
2. There is a metrizable compactum $X$ and a closed family $\mathcal{F} \subseteq \mathcal{K}(X)$ such that $\mathcal{F} \cong \mathcal{C}$.
3. There is a closed family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$.

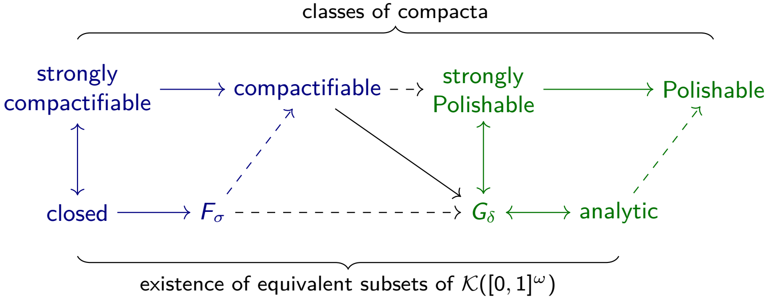
1. $\mathcal{C}$ is a strongly Polishable class of compacta.
2. There is a Polish space $X$ and an analytic family $\mathcal{F} \subseteq \mathcal{K}(X)$ such that $\mathcal{F} \cong \mathcal{C}$.
3. There is a $G_\delta$ family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$.
4. There is a closed family $\mathcal{F} \subseteq \mathcal{K}((0,1)^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$.

# Implications between the classes considered

## Proposition

Let $\mathcal{A}(q\colon A \to B)$ be a Polish composition of compacta.

- If $q$ is closed, then $\mathcal{F}_\mathcal{A} \subseteq \mathcal{K}(A)$ is $G_\delta$.
- Every compactifiable class is a strongly Polishable class.



# Borel complexity up to equivalence

The previous results show that the problem of compactifiability is related to the Borel complexity of subsets of $\mathcal{K}([0,1]^\omega))$ up to the equivalence.

- Closed subsets of $\mathcal{K}([0,1]^\omega)$ correspond to strongly compactifiable classes.
- To every analytic subset of $\mathcal{K}([0,1]^\omega)$ there exists an equivalent $G_\delta$ subset, and these correspond to strongly Polishable classes of compacta.

**Theorem [Kechris, Louveau, Woodin]**

Every analytic $\sigma$-ideal in $\mathcal{K}(X)$ for $X$ metrizable compact is $G_\delta$.

What about clopen, open, and $F_\sigma$ subsets of $\mathcal{K}([0,1]^\omega)$?

**Proposition**

There are only four clopen subsets of $\mathcal{K}([0,1]^\omega)$:

$$\emptyset,\ \ \{\emptyset\},\ \ \mathcal{K}([0,1]^\omega)\setminus\{\emptyset\},\ \ \mathcal{K}([0,1]^\omega).$$

# Open classes

Let $X$ be a metrizable compact space.

- $m(X) :=$ number of connected components of $X$.
- $n(X) :=$ number of nondegenerate components of $X$.
- $t(X) := (m(X), n(X))$ if $m(X) < \omega$, $\infty$ otherwise.
- $T := \{(m, n) : m \geq n \in \omega\}$, $T_+ := \{(m, n) \in T : m > 0\}$.
- We define a partial order $\leq$ on $T \cup \{\infty\}$:
  - $(0, 0)$ is not comparable with anything,
  - $T_+$ is ordered by the product order,
  - $\infty \geq t$ for every $t \in T_+$.
- For $t \in T \cup \{\infty\}$ we define the *principal upper class* $\mathcal{U}_t := \{X : t(X) \geq t\}$.

# Open classes

## Examples

We have the following classes of metrizable compact spaces:

- $\mathcal{U}_{0,0} = \{\emptyset\}$,
- $\mathcal{U}_{1,0}$ – all nonempty compacta,
- $\mathcal{U}_{2,0} \cup \mathcal{U}_{1,1}$ – all nondegenerate compacta,
- $\mathcal{U}_{m,0}$ – all compacta with at least $m$ components,
- $\mathcal{U}_{m,0} \cup \mathcal{U}_{1,1}$ – all compacta with at least $m$ points.

## Proposition

Let $X, Y \in \mathcal{K}([0,1]^\omega)$. A homeomorphic copy of $Y$ is contained in every neighborhood of $X$ if and only if $t(Y) \geq t(X)$.

# Open classes

- It follows that for every open $\mathcal{U} \subseteq \mathcal{K}([0,1]^\omega)$ we have $\mathcal{U} \cong \bigcup\{\mathcal{U}_{t(X)} : X \in \mathcal{U}\}$.
- For every $R \subseteq T$ let $A(R)$ denote the set of all $\leq$-minimal elements of $R$. This is a finite antichain in $T \cup \{\infty\}$. We have $\bigcup_{t \in R} \mathcal{U}_t = \bigcup_{t \in A(R)} \mathcal{U}_t$.
- Since finite spaces are dense in $\mathcal{K}([0,1]^\omega)$, not every upper class $\mathcal{U}_t$ is open in $\mathcal{K}([0,1]^\omega)$. On the other hand, this is essentially the only obstacle.
- We call a set $R \subseteq T$ *nice* if $(m, 0) \in R$ for some $m > 0$ whenever $R \cap T_+ \neq \emptyset$.
- $\bigcup_{t \in R} \mathcal{U}_t \cap \mathcal{K}([0,1]^\omega)$ is open if and only if $R$ is nice.
- $A(R)$ is nice if and only if $R$ is nice.

# Open classes

- We denote the set of all nice antichains in $T \cup \{\infty\}$ by $\mathcal{R}$. These are finite subsets of $T$.
- For every $R \in \mathcal{R}$ we define the *open class* $\mathcal{O}_R := \bigcup_{t \in R} \mathcal{U}_t$.

**Theorem**

- For every open $\mathcal{U} \subseteq \mathcal{K}([0,1]^\omega)$ there exists exactly one $R \in \mathcal{R}$ such that $\mathcal{U} \cong \mathcal{O}_R$.
- For every $R \in \mathcal{R}$ we have $\mathcal{O}_R \cong \mathcal{O}_R \cap \mathcal{K}([0,1]^\omega)$, which is open.

# $F_\sigma$ subsets

**Proposition**

Let $X \in \mathcal{K}([0,1]^\omega)$ and $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ closed. An equivalent copy of $\mathcal{F}$ is contained in every neighborhood of $X$ if and only if $t(Y) \geq t(X)$ for every $Y \in \mathcal{F}$.

**Theorem**

Every countable union of strongly compactifiable classes is strongly compactifiable, i.e. every $F_\sigma$ subset of $\mathcal{K}([0,1]^\omega)$ is strongly compactifiable and equivalent to a closed subset of $\mathcal{K}([0,1]^\omega)$.

The proof uses the notion of Z-sets from infinite-dimensional topology, a variant of Michael zero-dimensional selection theorem, and the previous classification of open subsets.

# Induced classes

Let $\mathcal{C}$ be a class of metrizable compacta.

- $\mathcal{C}^{\downarrow}$ denotes the class of all subspaces of members of $\mathcal{C}$ that are metrizable compact.

## Proposition

- If $\mathcal{C}$ is compactifiable, then $\mathcal{C}^{\downarrow}$ is strongly compactifiable.
- If $\mathcal{C}$ is Polishable, then $\mathcal{C}^{\downarrow}$ is strongly Polishable.

## Example

Every hereditary class of metrizable compacta with a universal element is strongly compactifiable – all compacta, all continua, continua with dimension at most $n$, chainable continua, tree-like continua, dendrites.

# Induced classes

Let $\mathcal{C}$ be a class of metrizable compacta.

- $\mathcal{C}^{\twoheadrightarrow}$ denotes the class of all continuous images of members of $\mathcal{C}$ that are metrizable compact.

**Proposition**

- If $\mathcal{C}$ is strongly Polishable, then $\mathcal{C}^{\twoheadrightarrow}$ is strongly Polishable.

**Example**

Every class of metrizable compacta closed under continuous images with a common model is strongly Polishable – Peano continua, weakly chainable continua.

# Induced classes

Let $\mathcal{C}$ be a class of metrizable compacta.

- $\mathcal{C}^{\uparrow}$ denotes the class of all superspaces of members of $\mathcal{C}$ that are metrizable compact.

## Proposition

- If $\mathcal{C}$ is strongly compactifiable, then $\mathcal{C}^{\uparrow}$ is strongly compactifiable.
- If $\mathcal{C}$ is strongly Polishable, then $\mathcal{C}^{\uparrow}$ is strongly Polishable.

## Example

The class of all uncountable compacta is strongly compactifiable.

# Induced classes

Let $\mathcal{C}$ be a class of metrizable compacta.

- $\mathcal{C}^{\cong}$ denotes the class of all homeomorphic copies of members of $\mathcal{C}$.

## Proposition

- If $\mathcal{C}$ strongly Polishable and $X$ is a Polish space, then $\mathcal{C}^{\cong} \cap \mathcal{K}(X)$ is analytic.

## Example

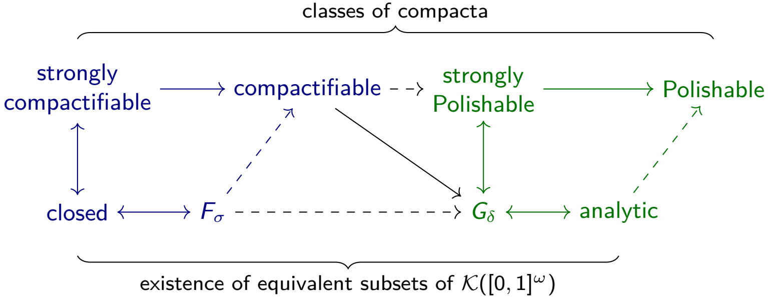
Classes coanalytically complete in $\mathcal{K}([0,1]^{\omega})$ are not strongly Polishable – all countable compacta, hereditarily decomposable continua, dendroids, $\lambda$-dendroids, arcwise connected continua, uniquely arcwise connected continua, hereditarily locally connected continua.

# Problems

## Questions

- Is there a compactifiable (or Polishable) class that is not strongly compactifiable (or strongly Polishable)?
- Is there a Polishable class that is not compactifiable?
- Is the class of all Peano continua compactifiable?

Thank you for your attention.




arXiv:1801.01826 (work in progress)