On semitopological locally compact graph inverse semigroups

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Definition

A semigroup S is called an *inverse semigroup* if for every a in S there exists an unique element a^{-1} in S such that

$$aa^{-1}a = a \qquad \text{and} \qquad a^{-1}aa^{-1} = a^{-1}.$$

Definition

For a non-zero cardinal λ, the polycyclic monoid on λ generators P_λ is the semigroup with zero given by the presentation:

$$P_\lambda = \left\langle \{p_i\}_{i\in\lambda}, \{p_i^{-1}\}_{i\in\lambda} \mid p_ip_i^{-1} = 1, p_ip_j^{-1} = 0 \text{ for } i \neq j \right\rangle.$$

Remark

1-Polycyclic monoid is isomorphic to the bicyclic semigroup with adjoint zero.

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Definition

For a given directed graph $E = (E^0, E^1, r, s)$ a graph inverse semigroup $G(E)$ over a graph E is a semigroup with zero generated by the sets E^0, E^1 together with a set $E^{-1} = \{e^{-1} : e \in E^1\}$ satisfying the following relations for all $a, b \in E^0$ and $e, f \in E^1$:

(i) $a \cdot b = a$ if $a = b$ and $a \cdot b = 0$ if $a \neq b$;

(ii) $s(e) \cdot e = e \cdot r(e) = e$;

(iii) $e^{-1} \cdot s(e) = r(e) \cdot e^{-1} = e^{-1}$;

(iv) $e^{-1} \cdot f = r(e)$ if $e = f$ and $e^{-1} \cdot f = 0$ if $e \neq f$.

Remark

For each non-zero cardinal λ, λ-polycyclic monoid is isomorphic to the graph inverse semigroup over a graph E which consists of one vertex and λ distinct loops.

Theorem (B. 2017)

Each graph inverse semigroup $G(E)$ embeds into a λ-polycyclic monoid, where $\lambda = |G(E)|$ if $|G(E)| > \omega$ and $\lambda = 2$ if $|G(E)| \leq \omega$.

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A locally compact monothetic topological group is either compact or discrete.

Theorem (Eberhart, Selden, 1969).

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Theorem (2017)

Discrete topology is the only locally compact semigroup topology on the graph inverse semigroup $G(E)$ if and only if graph E contains a finite amount of vertices and there does not exist a pair of vertices $e, f \in E^0$ such that the set $\{u \in \mathrm{Path}(E) : r(u) = e\}$ is finite and the set $\{a \in E^1 : s(a) = e \text{ and } r(a) = f\}$ is infinite.

Corollary (B., Gutik, 2016)

Locally compact topological λ-polycyclic monoid is the discrete space.

Proposition (2017)

There exists a non-discrete locally compact semigroup topology on a graph inverse semigroup $G(E)$ if and only if there exists a non-discrete topology τ such that $(G(E), \tau)$ is a locally compact metrizable topological inverse semigroup.

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Main results (semitopological case)

Theorem (2018)

Each non-zero element of an arbitrary semitopological graph inverse semigroup is an isolated point.

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Let E be a strongly connected graph which contains a finite amount of vertices. Then a locally compact semitopological graph inverse semigroup $G(E)$ is either compact or discrete.

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Main results (compact case)

Theorem (2018)

For a semitopological graph inverse semigroup $G(E)$ the following conditions are equivalent:

(1) $(G(E), \tau_{comp})$ is a topological semigroup;

(2) for each element $uv^{-1} \in G(E)$ the set $M_{uv^{-1}} = \{(ab^{-1}, cd^{-1}) \in G(E) \times G(E) \mid ab^{-1} \cdot cd^{-1} = uv^{-1}\}$ is finite;

(3) the indegree of each vertex of graph E is finite;

(4) Each $\mathcal{D}$-class in $G(E)$ is finite.

Theorem (2018)

For a semitopological graph inverse semigroup $G(E)$ the following conditions are equivalent:

(1) $G(E)$ embeds into a compact topological semigroup S;

(2) $G(E)$ embeds into a sequentially compact topological semigroup S;

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Natural question

Question

Let graph inverse semigroup $G(E)$ embeds into a countably compact topological semigroup S. Is it true that $G(E)$ is homeomorphic to $(G(E), \tau_{comp})$?

Answer

No! It follows from the following Theorem:

Theorem (Banakh, Dimitrova, Gutik, 2010)

If there is a torsion-free Abelian countably compact topological group G without non-trivial convergent sequences, then there exists a Tychonoff countably compact semigroup S containing a bicyclic semigroup.

Remark

The first example of a group G with properties required in the above Theorem was constructed by M. Tkachenko under the Continuum Hypothesis. Later, the Continuum Hypothesis was weakened to Martin's Axiom for σ-centered posets by A. Tomita, for countable posets by P. Koszmider, A. Tomita, S. Watson, and finally to the existence of continuum many incomparable selective ultrafilters by R. Madariaga-Garcia and A. Tomita. However, the problem of the existence of a countably compact group without convergent sequences in ZFC seems to be open.

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Thank You for attention!