The generalized meager ideal and clubs

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κ will always be a regular uncountable cardinal.

Definition

We endow 2^κ with the topology generated by basic open sets of the form $[s] = \{x \in 2^\kappa : s \subseteq x\}$ for $s \in 2^{<\kappa}$.

Definition

A set $X \subseteq 2^\kappa$ is called nowhere dense if
$\forall s \in 2^{<\kappa} \exists s' \in 2^{<\kappa} (s \subseteq s' \wedge [s'] \cap X = \emptyset)$.
$X \subseteq 2^\kappa$ is meager if it is the union of κ many nowhere dense sets.
The ideal of meager sets is denoted with $\mathcal{M}_\kappa$.

As usual we define cardinal characteristics related to this ideal:

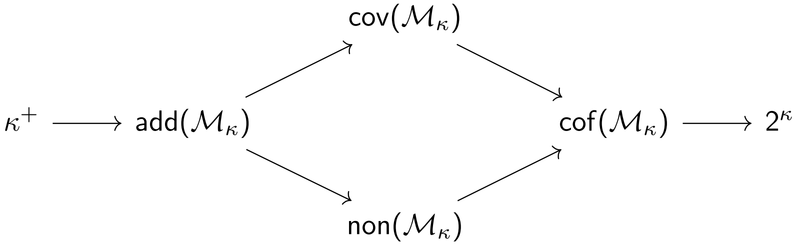
Definition

$$\mathrm{add}(\mathcal{M}_\kappa) = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{M}_\kappa \wedge \bigcup \mathcal{B} \notin \mathcal{M}_\kappa\}$$

$$\mathrm{cov}(\mathcal{M}_\kappa) = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{M}_\kappa \wedge \bigcup \mathcal{B} = 2^\kappa\}$$

$$\mathrm{non}(\mathcal{M}_\kappa) = \min\{|X| : X \subseteq 2^\kappa \wedge X \notin \mathcal{M}_\kappa\}$$

$$\mathrm{cof}(\mathcal{M}_\kappa) = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{M}_\kappa \wedge \mathcal{I}(\mathcal{B}) = \mathcal{M}_\kappa\}$$



One of the classical theorems that hold true at ω is:

Theorem

$\mathfrak{t} \leq \text{add}(\mathcal{M})$.

We generalized this theorem to κ.

Definition

A (κ-)tower is a sequence $\langle A_\alpha : \alpha < \delta \rangle$ maximal with the properties:

- $\forall \alpha < \lambda (A_\alpha \in [\kappa]^\kappa)$
- $\forall \alpha < \beta (A_\beta \subseteq^* A_\alpha)$
- $\forall I \in [\delta]^{<\kappa} (\bigcap_{\alpha \in I} A_\alpha \in [\kappa]^\kappa)$

The tower number $\mathfrak{t}(\kappa)$ is the least possible δ.

Theorem

Assume $\kappa^{<\kappa} = \kappa$, *then* $\mathfrak{t}(\kappa) \leq \mathrm{add}(\mathcal{M}_\kappa)$.

In order to prove the theorem we needed to introduce the following notion of club subset of $2^{<\kappa}$:

Definition

Let $C \subseteq 2^{<\kappa}$. Then we call C club iff:

- $\forall s \in 2^{<\kappa} \exists s' \supseteq s (s' \in C)$
- for every sequence $\langle s_i : i < \delta \rangle$ where $\delta < \kappa$, $s_i \in C$ for every $i < \delta$ and $s_i \subseteq s_j$ for $i < j$, $\bigcup_{i<\delta} s_i \in C$.

The intersection of less than κ many clubs is still club.

We write $C \subseteq^{**} D$ whenever $C \setminus 2^{<\alpha} \subseteq D$ for some $\alpha < \kappa$.

Sketch of proof of $\mathfrak{t}(\kappa) \leq \mathrm{add}(\mathcal{M}_\kappa)$.

Assume $\langle Y_\alpha : \alpha < \lambda \rangle$ are open dense sets in 2^κ and $\lambda < \mathfrak{t}(\kappa)$. We can write $Y_\alpha = \bigcup_{s \in S_\alpha} [s]$ where S_α is upwards closed.
We will construct a $\subseteq^{**}$ tower $\langle D_\alpha : \alpha < \lambda \rangle$ consisting of clubs on $2^{<\kappa}$, so that $D_\alpha \subseteq S_\alpha$ for each α.

- Start with $D_0 = S_0$.
- The successor steps are easy: $D_{\alpha+1} = D_\alpha \cap S_{\alpha+1}$.
- Limits of cofinality $< \kappa$ are easy by taking intersections.
- For limits of cofinality $\geq \kappa$ we use that we have a sequence of length $< \mathfrak{t}(\kappa)$ and some combinatorial tricks to get a club $\subseteq^{**}$ pseudointersection.

Finally find D a club $\subseteq^{**}$ of $\langle D_\alpha : \alpha < \lambda \rangle$. $Y := \bigcap_{i<\kappa} \bigcup_{\substack{s \in D, \\ \mathrm{lth}(s) \geq i}} [s]$ is comeager and subset of every Y_α.

□

The following is a very useful characterization of the bounding number:

Lemma

$\mathfrak{b}(\kappa) = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{C}, \mathcal{B}$ *has no pseudointersection*$\}$ *where* $\mathcal{C}$ *is the set of clubs on* κ.

Sketch of proof.

Basic idea:
For any $f \in \kappa^\kappa$, $C_f = \{\alpha < \kappa : f''\alpha \subseteq \alpha\}$ is club and $f \leq^* g$ implies $C_g \subseteq^* C_f$.
For any C club, $f_C(\alpha) := \min C \cap (\alpha, \kappa)$. $C \subseteq^* D$ implies $f_D \leq^* f_C$. □

This also gives a relatively easy proof of:

Theorem

$\mathfrak{t}(\kappa) \leq \mathfrak{b}(\kappa)$.

and the proof of $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$ becomes simple to explain:

Theorem (D. Raghavan, S. Shelah)

$\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$.

Proof.

Let $\mathcal{B}$ be an unbounded family of clubs and M and elementary submodel of size $|\mathcal{B}|$ containing $\mathcal{B}$.
Suppose $x \in [\kappa]^\kappa$ is unsplit over M. Then x generates an ultrafilter $\mathcal{U} = \{y \in M : x \subseteq^* y\}$ over M, κ-complete over M.
$\mathcal{U}$ can be "normalized" via a function f, i.e.
$\mathcal{V} = \{y \in M : f^{-1}(y) \in \mathcal{U}\}$ is a normal ultrafilter over M. Thus extending the club filter. But x induces a pseudointersection $(f''x)$ of $\mathcal{V} \supseteq \mathcal{B}$. □

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and dually:

Lemma

$\mathfrak{d}(\kappa) = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{C}, \mathcal{B}$ *is a base of* $\mathcal{C}\}$.

In this light it is natural to define:

Definition

$\mathfrak{p}_{2^{<\kappa}} = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{C}_{2^{<\kappa}}, \mathcal{B}$ has no $\subseteq^{**}$ pseudointersection $\}$ where $\mathcal{C}_{2^{<\kappa}}$ is the set of clubs on $2^{<\kappa}$.

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Lemma

If $\kappa^{<\kappa} = \kappa$, $\mathfrak{p}_{2^{<\kappa}} \leq \mathrm{add}(\mathcal{M}_\kappa)$.

Proof.

Given a $\langle Y_\alpha : \alpha < \lambda \rangle$ open dense, we find $\langle S_\alpha : \alpha < \lambda \rangle$ clubs so that $Y_\alpha = \bigcup_{s \in S_\alpha} [s]$ for every α.
If S is a club $\subseteq^{**}$ pseudointersection of $\langle S_\alpha : \alpha < \lambda \rangle$, then $Y := \bigcap_{i<\kappa} \bigcup_{\substack{s \in D, \\ \mathrm{lth}(s) \geq i}} [s]$ is comeager and subset of every Y_α. □

Definition

$\mathfrak{p}_{2^{<\kappa}} = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{C}_{2^{<\kappa}}, \mathcal{B}$ has no $\subseteq^{**}$ pseudointersection $\}$ where $\mathcal{C}_{2^{<\kappa}}$ is the set of clubs on $2^{<\kappa}$.

Lemma

If $\kappa^{<\kappa} = \kappa$, $\mathfrak{p}_{2^{<\kappa}} \leq \mathrm{add}(\mathcal{M}_\kappa)$.

Lemma

$\mathfrak{p}_{2^{<\kappa}} \leq \mathfrak{b}(\kappa)$.

Proof.

Given $\mathcal{B}$ a family of clubs on κ with no pseudointersection, we find that $\{\bigcup_{\alpha \in B} 2^\alpha : B \in \mathcal{B}\}$ has no $\subseteq^{**}$ pseudointersection. □

Theorem

Assume $\kappa^{<\kappa} = \kappa$, *then* $\mathfrak{p}_{2^{<\kappa}} = \min\{\mathfrak{b}(\kappa), \mathrm{cov}(\mathcal{M}_\kappa)\}$.

Proof.

Let $\{C_\alpha : \alpha < \lambda\}$ be a family of clubs on $2^{<\kappa}$ with $\lambda < \mathrm{cov}(\mathcal{M}_\kappa), \mathfrak{b}(\kappa)$.
Consider the sets $Y_\alpha = \bigcap_{i<\kappa} \bigcup_{s \in C_\alpha, s \notin 2^{<i}} [s]$ and let $Y := \bigcap_{\alpha<\lambda} Y_\alpha$.
We find a dense subset $\{x_i : i < \kappa\} \subseteq Y$.
Note that for every $i < \kappa$ and every $\alpha < \lambda$, the set $C^i_\alpha = \{j < \kappa : x_i \restriction j \in C_\alpha\}$ is a club on κ.
As $\lambda < \mathfrak{b}(\kappa)$, for each $i < \kappa$, $\mathcal{B}_i = \{C^i_\alpha : \alpha < \lambda\}$ has a pseudointersection $B_i \in [\kappa]^\kappa$.
Again applying $\lambda < \mathfrak{b}(\kappa)$ we can find a function $f \in \kappa^\kappa$ so that

$$\forall \alpha < \lambda(|\kappa \setminus \{i \in \kappa : B_i \setminus 2^{<f(i)} \subseteq C^i_\alpha\}| < \kappa).$$

Proof.

Now enumerate $2^{<\kappa}$ as $\langle s_i : i < \kappa \rangle$ and for every i find $\sigma_i \supseteq s_i$ so that $\sigma_i \in B_j \setminus 2^{<f(j)}$ for some $j > i$.
The collection $C' = \{\sigma_i : i \in \kappa\}$ is unbounded in $2^{<\kappa}$. Furthermore we have that $C' \subseteq^* C_\alpha$ for every α. If C is the closure of C', then $C \subseteq^{**} C_\alpha$ for every α.
Thus we have shown that $\{C_\alpha : \alpha < \lambda\}$ has a $\subseteq^{**}$ pseudointersection which is club.

□

Theorem

Assume $\kappa^{<\kappa} = \kappa$, *then* $\mathfrak{p}_{2^{<\kappa}} = \min\{\mathfrak{b}(\kappa), \mathrm{cov}(\mathcal{M}_\kappa)\}$.

By an unpublished result of J. Brendle, we have that $\mathrm{add}(\mathcal{M}_\kappa) \leq \mathfrak{b}(\kappa)$ (which was previously only known for κ inaccessible). From this we get the following corollary.

Corollary

$\mathfrak{p}_{2^{<\kappa}} = \mathrm{add}(\mathcal{M}_\kappa)$.

Proof.

If $\kappa^{<\kappa} = \kappa$, then $\min\{\mathfrak{b}(\kappa), \mathrm{cov}(\mathcal{M}_\kappa)\} = \mathfrak{p}_{2^{<\kappa}} \leq \mathrm{add}(\mathcal{M}_\kappa)$ $\leq \mathfrak{b}(\kappa), \mathrm{cov}(\mathcal{M}_\kappa)$.
If $\kappa^{<\kappa} > \kappa$, then $\kappa^+ \leq \mathfrak{p}_{2^{<\kappa}} \leq \mathrm{add}(\mathcal{M}_\kappa) = \kappa^+$. □

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$\mathfrak{p}_{2^{<\kappa}} = \mathrm{add}(\mathcal{M}_\kappa)$.

But the following seems to be open:

Question

Is $\mathrm{add}(\mathcal{M}_\kappa) < \mathfrak{b}(\kappa)$ *consistent (possibly assuming LC)? I.e. is* $\mathfrak{p}_{2^{<\kappa}} < \mathfrak{b}(\kappa)$ *consistent?*

Question

How much are clubs on κ *and on* $2^{<\kappa}$ *related?*

Thanks for your attention!