Triviality and nontriviality of homeomorphisms of Čech-Stone remainders

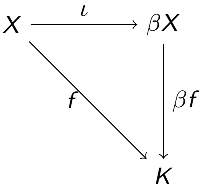
Alessandro Vignati
IMJ-PRG - Université Paris Diderot

Winter School in Abstract Analysis
section Set Theory and Topology
Hejnice, 27 January 2018

X always denotes a second-countable locally compact Hausdorff space.

X always denotes a second-countable locally compact Hausdorff space. βX is its Čech-Stone compactification that is "the largest compact space in which X is dense and open" or

X always denotes a second-countable locally compact Hausdorff space. βX is its Čech-Stone compactification that is "the largest compact space in which X is dense and open" or the compact Hausdorff space satisfying the following universal property: there is a continuous injection $\iota\colon X \to \beta X$ such that whenever K is compact Hausdorff and $f\colon X \to K$ is continuous there is a unique continuous map $\beta f\colon \beta X \to K$ such that



X always denotes a second-countable locally compact Hausdorff space. βX is its Čech-Stone compactification that is "the largest compact space in which X is dense and open" or the compact Hausdorff space satisfying the following universal property: there is a continuous injection $\iota\colon X \to \beta X$ such that whenever K is compact Hausdorff and $f\colon X \to K$ is continuous there is a unique continuous map $\beta f\colon \beta X \to K$ such that

$$\begin{array}{ccc} X & \xrightarrow{\iota} & \beta X \\ & \searrow^{f} & \downarrow{\beta f} \\ & & K \end{array}$$

We will omit ι, and identify X with $\iota[X]$. The space $X^* = \beta X \setminus X$ is the Čech-Stone remainder of X.

- If X is just Tychonoff such a ι still exists. In fact $\iota[X]$ is open in βX if and only if X is locally compact.

- If X is just Tychonoff such a ι still exists. In fact $\iota[X]$ is open in βX if and only if X is locally compact.
- βX is unique, and can be identified with the set of all ultrafilters on the zero-sets of X. In this case X is identified with the principal ultrafilters;

- If X is just Tychonoff such a ι still exists. In fact $\iota[X]$ is open in βX if and only if X is locally compact.
- βX is unique, and can be identified with the set of all ultrafilters on the zero-sets of X. In this case X is identified with the principal ultrafilters;
- Exercise: βX is never second-countable unless X is compact (then $X = \beta X$).

- If X is just Tychonoff such a ι still exists. In fact $\iota[X]$ is open in βX if and only if X is locally compact.
- βX is unique, and can be identified with the set of all ultrafilters on the zero-sets of X. In this case X is identified with the principal ultrafilters;
- Exercise: βX is never second-countable unless X is compact (then $X = \beta X$).
- If X is second-countable and noncompact, X^* is not second-countable. Second-countability of X is needed (e.g., $X = \omega_1$..).

- If X is just Tychonoff such a ι still exists. In fact $\iota[X]$ is open in βX if and only if X is locally compact.
- βX is unique, and can be identified with the set of all ultrafilters on the zero-sets of X. In this case X is identified with the principal ultrafilters;
- Exercise: βX is never second-countable unless X is compact (then $X = \beta X$).
- If X is second-countable and noncompact, X^* is not second-countable. Second-countability of X is needed (e.g., $X = \omega_1$..).
- If X is locally compact, there are three C^*-algebras associated to X: $C_0(X)$ (=cnts functions vanishing at infinity), $C_b(X)$ (=cnts bounded functions) and $C(X^*)$ (=cnts functions on X^*).

- If X is just Tychonoff such a ι still exists. In fact $\iota[X]$ is open in βX if and only if X is locally compact.
- βX is unique, and can be identified with the set of all ultrafilters on the zero-sets of X. In this case X is identified with the principal ultrafilters;
- Exercise: βX is never second-countable unless X is compact (then $X = \beta X$).
- If X is second-countable and noncompact, X^* is not second-countable. Second-countability of X is needed (e.g., $X = \omega_1$..).
- If X is locally compact, there are three C^*-algebras associated to X: $C_0(X)$ (=cnts functions vanishing at infinity), $C_b(X)$ (=cnts bounded functions) and $C(X^*)$ (=cnts functions on X^*). In this case, homeomorphisms of X^* are automorphisms of $C(X^*)$ and there is a canonical quotient map $\pi_X \colon C_b(X) \to C(X^*) = C_b(X)/C_0(X)$

- If X is just Tychonoff such a ι still exists. In fact $\iota[X]$ is open in βX if and only if X is locally compact.
- βX is unique, and can be identified with the set of all ultrafilters on the zero-sets of X. In this case X is identified with the principal ultrafilters;
- Exercise: βX is never second-countable unless X is compact (then $X = \beta X$).
- If X is second-countable and noncompact, X^* is not second-countable. Second-countability of X is needed (e.g., $X = \omega_1$..).
- If X is locally compact, there are three C^*-algebras associated to X: $C_0(X)$ (=cnts functions vanishing at infinity), $C_b(X)$ (=cnts bounded functions) and $C(X^*)$ (=cnts functions on X^*). In this case, homeomorphisms of X^* are automorphisms of $C(X^*)$ and there is a canonical quotient map $\pi_X\colon C_b(X) \to C(X^*) = C_b(X)/C_0(X)$
- If $\phi \in \mathrm{Homeo}(X^*)$ and $\tilde{\phi} \in \mathrm{Aut}(C(X^*))$ is the dual isomorphism we say that $\tilde{\Phi}\colon C_b(X) \to C_b(X)$ is a lifting for $\tilde{\phi}$ if it lifts:

$$\pi_X(\tilde{\Phi}(a)) = \tilde{\phi}(\pi_X(a)),\ a \in C_b(X).$$

We study the homeomorphisms of such spaces. Let's start with $\mathbb{N}$.

We study the homeomorphisms of such spaces. Let's start with $\mathbb{N}$. It is easy to see that all homeomorphisms of $\beta\mathbb{N}$ come from permutations of $\mathbb{N}$. It is not that easy to describe homeomorphisms of $\mathbb{N}^*$.

We study the homeomorphisms of such spaces. Let's start with $\mathbb{N}$. It is easy to see that all homeomorphisms of $\beta\mathbb{N}$ come from permutations of $\mathbb{N}$. It is not that easy to describe homeomorphisms of $\mathbb{N}^*$.

Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a "permutation modulo finite" that is, there are finite $F_1, F_2 \subset \mathbb{N}$ such that $f \restriction \mathbb{N} \setminus F_1 \to \mathbb{N} \setminus F_2$ is a bijection. Then $\beta f \restriction \mathbb{N}^*$ is an homeomorphism.

We study the homeomorphisms of such spaces. Let's start with $\mathbb{N}$. It is easy to see that all homeomorphisms of $\beta\mathbb{N}$ come from permutations of $\mathbb{N}$. It is not that easy to describe homeomorphisms of $\mathbb{N}^*$.

Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a "permutation modulo finite" that is, there are finite $F_1, F_2 \subset \mathbb{N}$ such that $f \restriction \mathbb{N} \setminus F_1 \to \mathbb{N} \setminus F_2$ is a bijection. Then $\beta f \restriction \mathbb{N}^*$ is an homeomorphism.

Definition

$\phi \in \text{Homeo}(\mathbb{N}^*)$ is said **trivial** if there is an almost permutation such that $\phi(x) = \{f[A] \mid A \in x\}$ for all $x \in \mathbb{N}^*$. (equivalently $\phi = \beta f \restriction \mathbb{N}^*$).

We study the homeomorphisms of such spaces. Let's start with $\mathbb{N}$. It is easy to see that all homeomorphisms of $\beta\mathbb{N}$ come from permutations of $\mathbb{N}$. It is not that easy to describe homeomorphisms of $\mathbb{N}^*$.

Suppose that $f\colon \mathbb{N} \to \mathbb{N}$ is a "permutation modulo finite" that is, there are finite $F_1, F_2 \subset \mathbb{N}$ such that $f \restriction \mathbb{N} \setminus F_1 \to \mathbb{N} \setminus F_2$ is a bijection. Then $\beta f \restriction \mathbb{N}^*$ is an homeomorphism.

Definition

$\phi \in \mathrm{Homeo}(\mathbb{N}^*)$ is said **trivial** if there is an almost permutation such that $\phi(x) = \{f[A] \mid A \in x\}$ for all $x \in \mathbb{N}^*$. (equivalently $\phi = \beta f \restriction \mathbb{N}^*$).

Note that there are only $\mathfrak{c}$ trivial homeomorphisms.

We study the homeomorphisms of such spaces. Let's start with $\mathbb{N}$. It is easy to see that all homeomorphisms of $\beta\mathbb{N}$ come from permutations of $\mathbb{N}$. It is not that easy to describe homeomorphisms of $\mathbb{N}^*$.

Suppose that $f\colon \mathbb{N} \to \mathbb{N}$ is a "permutation modulo finite" that is, there are finite $F_1, F_2 \subset \mathbb{N}$ such that $f \restriction \mathbb{N} \setminus F_1 \to \mathbb{N} \setminus F_2$ is a bijection. Then $\beta f \restriction \mathbb{N}^*$ is an homeomorphism.

Definition

$\phi \in \mathrm{Homeo}(\mathbb{N}^*)$ is said **trivial** if there is an almost permutation such that $\phi(x) = \{f[A] \mid A \in x\}$ for all $x \in \mathbb{N}^*$. (equivalently $\phi = \beta f \restriction \mathbb{N}^*$).

Note that there are only $\mathfrak{c}$ trivial homeomorphisms.

Question

Are all homeomorphisms of $\mathbb{N}^*$ trivial?

Question

Are all homeomorphisms of $\mathbb{N}^*$ *trivial?*

Question

Are all homeomorphisms of $\mathbb{N}^*$ trivial?

Theorem

- *(Rudin) Assume CH. Then there are nontrivial homeomorphisms of $\mathbb{N}^*$. In fact there are $2^{\aleph_1} > \mathfrak{c}$ homeomorphisms.*

Question

Are all homeomorphisms of $\mathbb{N}^*$ *trivial?*

Theorem

- *(Rudin) Assume CH. Then there are nontrivial homeomorphisms of* $\mathbb{N}^*$. *In fact there are* $2^{\aleph_1} > \mathfrak{c}$ *homeomorphisms.*
- *(Shelah, Shelah-Steprans, Velickovic) It is consistent that all homeomorphisms of* $\mathbb{N}^*$ *are trivial. In fact, it follows from Forcing Axioms.*

Question

Are all homeomorphisms of $\mathbb{N}^*$ trivial?

Theorem

- *(Rudin) Assume CH. Then there are nontrivial homeomorphisms of $\mathbb{N}^*$. In fact there are $2^{\aleph_1} > \mathfrak{c}$ homeomorphisms.*
- *(Shelah, Shelah-Steprans, Velickovic) It is consistent that all homeomorphisms of $\mathbb{N}^*$ are trivial. In fact, it follows from Forcing Axioms.*

History fact: Rudin ('56) wasn't trying to prove the existence of nontrivial homeomorphisms. He was in fact trying to show that $\mathbb{N}^*$ was not homogeneous, and used CH to do so, constructing at the same time $2^{\aleph_1}$ homeomorphisms of $\mathbb{N}^*$.

Question

Are all homeomorphisms of $\mathbb{N}^*$ trivial?

Theorem

- *(Rudin) Assume CH. Then there are nontrivial homeomorphisms of $\mathbb{N}^*$. In fact there are $2^{\aleph_1} > \mathfrak{c}$ homeomorphisms.*
- *(Shelah, Shelah-Steprans, Velickovic) It is consistent that all homeomorphisms of $\mathbb{N}^*$ are trivial. In fact, it follows from Forcing Axioms.*

History fact: Rudin ('56) wasn't trying to prove the existence of nontrivial homeomorphisms. He was in fact trying to show that $\mathbb{N}^*$ was not homogeneous, and used CH to do so, constructing at the same time $2^{\aleph_1}$ homeomorphisms of $\mathbb{N}^*$.

History fact 2: Shelah's proof ('82) is done with Forcing. Shelah-Steprans ('88) assumed PFA, while Velickovic ('93) showed that Todorcevic's OCA and Martin's Axiom (both consequences of PFA) are enough.

Definition

An homeomorphism ϕ of X^* is trivial if there are compact sets $K_1, K_2 \subseteq X$ and an homeomorphism $f \colon X \setminus K_1 \to X \setminus K_2$ such that $\phi(x) = \{f(C) \mid C \in x\}$.

Definition

An homeomorphism ϕ of X^* is trivial if there are compact sets $K_1, K_2 \subseteq X$ and an homeomorphism $f\colon X \setminus K_1 \to X \setminus K_2$ such that $\phi(x) = \{f(C) \mid C \in x\}$.

Question

Are all homeomorphisms of X^* trivial?

Definition

An homeomorphism ϕ of X^* is trivial if there are compact sets $K_1, K_2 \subseteq X$ and an homeomorphism $f\colon X \setminus K_1 \to X \setminus K_2$ such that $\phi(x) = \{f(C) \mid C \in x\}$.

Question

Are all homeomorphisms of X^* trivial?

Exercise: there are only $\mathfrak{c}$ trivial homeomorphisms.

Definition

An homeomorphism ϕ of X^* is trivial if there are compact sets $K_1, K_2 \subseteq X$ and an homeomorphism $f\colon X \setminus K_1 \to X \setminus K_2$ such that $\phi(x) = \{f(C) \mid C \in x\}$.

Question

Are all homeomorphisms of X^* trivial?

Exercise: there are only $\mathfrak{c}$ trivial homeomorphisms.

Conjecture

Suppose X is noncompact.

- *CH implies there are nontrivial homeomorphisms of X^*;*
- *PFA (or less) implies all homeomorphisms of X^* are trivial.*

If X is zero-dimensional then

- X^* is zero-dimensional and has no isolated points. Also, it has weight $\mathfrak{c}$

If X is zero-dimensional then

- X^* is zero-dimensional and has no isolated points. Also, it has weight $\mathfrak{c}$
- all nonempty G_δ subsets of X^* have nonempty interior

If X is zero-dimensional then

- X^* is zero-dimensional and has no isolated points. Also, it has weight $\mathfrak{c}$
- all nonempty G_δ subsets of X^* have nonempty interior
- all pairs of disjoint open F_σ subsets of X^* have disjoint closure

If X is zero-dimensional then

- X^* is zero-dimensional and has no isolated points. Also, it has weight $\mathfrak{c}$
- all nonempty G_δ subsets of X^* have nonempty interior
- all pairs of disjoint open F_σ subsets of X^* have disjoint closure

Theorem (Second Parovicenko's Theorem)

Assume CH. Every space with such properties is homeomorphic to $\mathbb{N}^*$. So there are $2^{\mathfrak{c}}$ homeomorphisms of X^*, and hence nontrivial ones.

	$CH \Rightarrow \exists$ nontrivial	Forcing Axioms $\Rightarrow$ all trivial
$X = \mathbb{N}$	Rudin	Velickovic
$dim(X) = 0$	Parovicenko	

CH, the half-open interval

$$CL(X) = \{a = \overline{a} \subseteq X\}$$

$$CL(X) = \{a = \overline{a} \subseteq X\}$$

For $a \in CL(X)$, let $a^* = \beta a \setminus a$ the correspondent closed set in X^*.

$$CL(X) = \{a = \overline{a} \subseteq X\}$$

For $a \in CL(X)$, let $a^* = \beta a \setminus a$ the correspondent closed set in X^*.

Definition

An homeomorphism ϕ of X^* has a **representation** if for all $a \in CL(X)$ there is $b \in CL(X)$ such that $\phi[a^*] = b^*$.

$$CL(X) = \{a = \overline{a} \subseteq X\}$$

For $a \in CL(X)$, let $a^* = \beta a \setminus a$ the correspondent closed set in X^*.

Definition

An homeomorphism ϕ of X^* has a **representation** if for all $a \in CL(X)$ there is $b \in CL(X)$ such that $\phi[a^*] = b^*$.

Proposition

If ϕ is trivial, ϕ has a representation.

$$CL(X) = \{a = \overline{a} \subseteq X\}$$

For $a \in CL(X)$, let $a^* = \beta a \setminus a$ the correspondent closed set in X^*.

Definition

An homeomorphism ϕ of X^* has a **representation** if for all $a \in CL(X)$ there is $b \in CL(X)$ such that $\phi[a^*] = b^*$.

Proposition

If ϕ is trivial, ϕ has a representation.

Theorem (Yu)

Assume CH. There is an homeomorphism of $[0,1)^*$ with no representation.

This was later refined (see K.P. Hart's work) to construct $2^{\mathfrak{c}}$ homeomorphisms of $[0,1)^*$ (again, under CH).

Recall:

- $C_b(X) \cong C(\beta X)$;
- $C_b(X)/C_0(X) \cong C(X^*)$;
- Homeomorphisms of X^* correspond to automorphisms of $C(X^*)$.

CH, other known cases

Recall:

- $C_b(X) \cong C(\beta X)$;
- $C_b(X)/C_0(X) \cong C(X^*)$;
- Homeomorphisms of X^* correspond to automorphisms of $C(X^*)$.

These objects have their own model theory associated in the setting of continuous model theory for C^*-algebras. In this setting, under CH, countably saturated objects of density $\mathfrak{c}$ have $2^{\mathfrak{c}}$ automorphisms.

Recall:

- $C_b(X) \cong C(\beta X)$;
- $C_b(X)/C_0(X) \cong C(X^*)$;
- Homeomorphisms of X^* correspond to automorphisms of $C(X^*)$.

These objects have their own model theory associated in the setting of continuous model theory for C^*-algebras. In this setting, under CH, countably saturated objects of density $\mathfrak{c}$ have $2^{\mathfrak{c}}$ automorphisms.

Theorem (Farah-Shelah)

If X_i, $i \in \mathbb{N}$, are compact second-countable spaces and $X = \sqcup X_i$ then $C(X^*)$ is countably saturated. So, under CH, X^* has nontrivial homeomorphisms.

Recall:

- $C_b(X) \cong C(\beta X)$;
- $C_b(X)/C_0(X) \cong C(X^*)$;
- Homeomorphisms of X^* correspond to automorphisms of $C(X^*)$.

These objects have their own model theory associated in the setting of continuous model theory for C^*-algebras. In this setting, under CH, countably saturated objects of density $\mathfrak{c}$ have $2^{\mathfrak{c}}$ automorphisms.

Theorem (Farah-Shelah)

If X_i, $i \in \mathbb{N}$, are compact second-countable spaces and $X = \sqcup X_i$ then $C(X^*)$ is countably saturated. So, under CH, X^* has nontrivial homeomorphisms.

This is the nicest existing proof, but the existence of nontrivial homeomorphisms of X^* for $X = \sqcup X_i$ was originally obtained by Coskey and Farah.

	$CH \Rightarrow \exists$ nontrivial	Forcing Axioms $\Rightarrow$ all trivial
$X = \mathbb{N}$	Rudin	Velickovic
$dim(X) = 0$	Parovicenko	
$X = [0,1)$, $X = \mathbb{R}$	Yu (but see K.P. Hart)	
$X = \bigsqcup X_i$, X_i cpct	Coskey-Farah, Farah-Shelah	

The most natural instances of higher dimensional connected spaces for which is was unknown whether CH implies the existence of nontrivial homeomorphisms were given by $X = \mathbb{R}^n$, for $n \geq 2$.

The most natural instances of higher dimensional connected spaces for which is was unknown whether CH implies the existence of nontrivial homeomorphisms were given by $X = \mathbb{R}^n$, for $n \geq 2$.

Theorem (V., '16)

Let X be a second-countable noncompact manifold and assume CH. Then X^* has nontrivial homeomorphisms.

The most natural instances of higher dimensional connected spaces for which is was unknown whether CH implies the existence of nontrivial homeomorphisms were given by $X = \mathbb{R}^n$, for $n \geq 2$.

Theorem (V., '16)

Let X be a second-countable noncompact manifold and assume CH. Then X^* has nontrivial homeomorphisms.

Sketch of the proof: (for $\mathbb{R}^n$). Fix open sets $U_i = \{x \mid d(x, (0, i)) < 1/3\}$. From this, for $f \in \mathbb{N}^{\mathbb{N}\uparrow}$ we can define

- C_f, a subalgebra of $C(X^*)$ and
- ϕ_f, an homeomorphism of X^*

such that

- $C(X^*) = \bigcup C_f$;
- $f \leq^* g$ implies $C_f \subseteq C_g$
- ϕ_f is the identity on C_f
- if $\forall^\infty n(nf(n) \leq g(n))$, there is $a \in C_g$ such that $\phi_f(a) \neq a$.

Sketch of the proof: (for $\mathbb{R}^n$, continued): Note that $\mathfrak{d} = \omega_1$, and fix a cofinal sequence, in $(\mathbb{N}^\mathbb{N}, \leq^*)$, of increasing functions such that $nf_\alpha(n) \leq f_{\alpha+1}(n)$ for all $\alpha < \omega_1$.

Sketch of the proof: (for $\mathbb{R}^n$, continued): Note that $\mathfrak{d} = \omega_1$, and fix a cofinal sequence, in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, of increasing functions such that $nf_\alpha(n) \leq f_{\alpha+1}(n)$ for all $\alpha < \omega_1$. Since we were careful enough in choosing the C_f's and the ϕ_f's, and we only have to care about countable steps, for all $p \in 2^{\omega_1}$ we can define an homeomorphism ϕ_p of X^*, with the property that $p \neq q \Rightarrow \phi_p \neq \phi_q$.

Sketch of the proof: (for $\mathbb{R}^n$, continued): Note that $\mathfrak{d} = \omega_1$, and fix a cofinal sequence, in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, of increasing functions such that $nf_\alpha(n) \leq f_{\alpha+1}(n)$ for all $\alpha < \omega_1$. Since we were careful enough in choosing the C_f's and the ϕ_f's, and we only have to care about countable steps, for all $p \in 2^{\omega_1}$ we can define an homeomorphism ϕ_p of X^*, with the property that $p \neq q \Rightarrow \phi_p \neq \phi_q$. Apply weak CH ($2^{\aleph_0} < 2^{\aleph_1}$).

Sketch of the proof: (for $\mathbb{R}^n$, continued): Note that $\mathfrak{d} = \omega_1$, and fix a cofinal sequence, in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, of increasing functions such that $nf_\alpha(n) \leq f_{\alpha+1}(n)$ for all $\alpha < \omega_1$. Since we were careful enough in choosing the C_f's and the ϕ_f's, and we only have to care about countable steps, for all $p \in 2^{\omega_1}$ we can define an homeomorphism ϕ_p of X^*, with the property that $p \neq q \Rightarrow \phi_p \neq \phi_q$. Apply weak CH ($2^{\aleph_0} < 2^{\aleph_1}$).

Theorem (V., '16)

Let X be a second-countable noncompact manifold and assume CH. Then X^* has nontrivial homeomorphisms.

Sketch of the proof: (for $\mathbb{R}^n$, continued): Note that $\mathfrak{d} = \omega_1$, and fix a cofinal sequence, in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, of increasing functions such that $nf_\alpha(n) \leq f_{\alpha+1}(n)$ for all $\alpha < \omega_1$. Since we were careful enough in choosing the C_f's and the ϕ_f's, and we only have to care about countable steps, for all $p \in 2^{\omega_1}$ we can define an homeomorphism ϕ_p of X^*, with the property that $p \neq q \Rightarrow \phi_p \neq \phi_q$. Apply weak CH ($2^{\aleph_0} < 2^{\aleph_1}$).

Theorem (V., '16)

Let X be a second-countable noncompact manifold and assume CH. Then X^* has nontrivial homeomorphisms.

	$CH \Rightarrow \exists$ nontrivial	Forcing Axioms $\Rightarrow$ all trivial
$X = \mathbb{N}$	Rudin	Velickovic
$dim(X) = 0$	Parovicenko	
$X = [0,1)$, $X = \mathbb{R}$	Yu (but see K.P. Hart)	
$X = \bigsqcup X_i$, X_i cpct	Coskey-Farah, Farah-Shelah	
X manifold	V.	

Conjecture

Assume Forcing Axiom. Then all homeomorphisms of X^* are trivial.

Conjecture

Assume Forcing Axiom. Then all homeomorphisms of X^* are trivial.

The first attempts to generalize Shelah's intuition was done by Farah.

Conjecture

Assume Forcing Axiom. Then all homeomorphisms of X^* are trivial.

The first attempts to generalize Shelah's intuition was done by Farah. He attacked the question on when $\alpha^*(=\beta\alpha\setminus\alpha)$ and β^* could be isomorphic, for countable ordinals $\alpha\neq\beta$.

PFA, the low dimensional case

Conjecture

Assume Forcing Axiom. Then all homeomorphisms of X^* are trivial.

The first attempts to generalize Shelah's intuition was done by Farah. He attacked the question on when $\alpha^*(=\beta\alpha\setminus\alpha)$ and β^* could be isomorphic, for countable ordinals $\alpha\neq\beta$. If $\phi\colon\alpha^*\to\beta^*$ is an homeomorphism, he proved, under Forcing Axioms, the existence of a well behaved lifting of the form:

$$\begin{array}{ccc} C_b(\beta) & \xrightarrow{\Phi} & C_b(\alpha) \\ \downarrow & & \downarrow \\ C(\beta^*) & \underset{\tilde{\phi}}{\longrightarrow} & C(\alpha^*) \end{array}$$

Conjecture

Assume Forcing Axiom. Then all homeomorphisms of X^* are trivial.

The first attempts to generalize Shelah's intuition was done by Farah. He attacked the question on when $\alpha^*(=\beta\alpha\setminus\alpha)$ and β^* could be isomorphic, for countable ordinals $\alpha\neq\beta$. If $\phi\colon\alpha^*\to\beta^*$ is an homeomorphism, he proved, under Forcing Axioms, the existence of a well behaved lifting of the form:

$$\begin{array}{ccc} C_b(\beta) & \xrightarrow{\Phi} & C_b(\alpha) \\ \downarrow & & \downarrow \\ C(\beta^*) & \xrightarrow[\tilde{\phi}]{} & C(\alpha^*) \end{array}$$

and used it to get

Theorem (Farah)

Assume Forcing Axioms and let $\alpha<\omega_1$ be a limit. Then all homeomorphisms of α^* are trivial. Also if α^* is homeomorphic to β^* then $\alpha=\beta$.

PFA and representations

Definition

An homeomorphism ϕ of X^* has a **representation** if for all $a \in CL(X)$ there is $b \in CL(X)$ such that $\phi[a^*] = b^*$.

Definition

An homeomorphism ϕ of X^* has a **representation** if for all $a \in CL(X)$ there is $b \in CL(X)$ such that $\phi[a^*] = b^*$.

Theorem (Farah-Shelah, '15)

Let X be second countable and assume Forcing Axioms. Let $\phi \in \mathrm{Homeo}(X^*)$ such that both ϕ and ϕ^{-1} have a representation. Then ϕ is trivial.

PFA and representations

Definition

An homeomorphism ϕ of X^* has a **representation** if for all $a \in CL(X)$ there is $b \in CL(X)$ such that $\phi[a^*] = b^*$.

Theorem (Farah-Shelah, '15)

Let X be second countable and assume Forcing Axioms. Let $\phi \in \mathrm{Homeo}(X^*)$ such that both ϕ and ϕ^{-1} have a representation. Then ϕ is trivial.

Theorem (Farah-McKenney, '12)

Let X be 0-dimensional and second countable and assume Forcing Axioms. Then all homeomorphisms of X^* are trivial.

Definition

An homeomorphism ϕ of X^* has a **representation** if for all $a \in CL(X)$ there is $b \in CL(X)$ such that $\phi[a^*] = b^*$.

Theorem (Farah-Shelah, '15)

Let X be second countable and assume Forcing Axioms. Let $\phi \in \mathrm{Homeo}(X^*)$ such that both ϕ and ϕ^{-1} have a representation. Then ϕ is trivial.

Theorem (Farah-McKenney, '12)

Let X be 0-dimensional and second countable and assume Forcing Axioms. Then all homeomorphisms of X^* are trivial.

	$CH \Rightarrow \exists$ nontrivial	Forcing Axioms $\Rightarrow$ all trivial
$X = \mathbb{N}$	Rudin	Velickovic
$dim(X) = 0$	Parovicenko	Farah, Farah-McKenney
$X = [0,1)$, $X = \mathbb{R}$	Yu (but see K.P. Hart)	
$X = \bigsqcup X_i$, X_i cpct	Coskey-Farah, Farah-Shelah	
X manifold	V.	

Let $D(X) = \{a = \overline{a} \subseteq X \mid a \text{ is countable and discrete}\}$.

Definition

An homeomorphism ϕ of X^* has a **local representation** if for all $a \in D(X)$ there is $b \in D(X)$ such that $\phi[a^*] = b^*$.

Let $D(X) = \{a = \overline{a} \subseteq X \mid a \text{ is countable and discrete}\}$.

Definition

An homeomorphism ϕ of X^* has a **local representation** if for all $a \in D(X)$ there is $b \in D(X)$ such that $\phi[a^*] = b^*$.

Lemma

Assume Forcing Axioms, let X be second countable and suppose that $\phi \in \mathrm{Homeo}(X^*)$. If both ϕ and ϕ^{-1} have a local representation, then they have a representation.

Let $D(X) = \{a = \overline{a} \subseteq X \mid a \text{ is countable and discrete}\}$.

Definition

An homeomorphism ϕ of X^* has a **local representation** if for all $a \in D(X)$ there is $b \in D(X)$ such that $\phi[a^*] = b^*$.

Lemma

Assume Forcing Axioms, let X be second countable and suppose that $\phi \in \mathrm{Homeo}(X^*)$. If both ϕ and ϕ^{-1} have a local representation, then they have a representation.

(This proof is a modification of Farah and Shelah's proof. Nothing fancy here.)

Theorem (V.,'17)

Assume Forcing Axioms and suppose that X is connected and second-countable. Then all homeomorphisms of X^* have a local representation.

Theorem (V.,'17)

Assume Forcing Axioms and suppose that X is connected and second-countable. Then all homeomorphisms of X^* have a local representation. So they're all trivial.

Theorem (V.,'17)

Assume Forcing Axioms and suppose that X is connected and second-countable. Then all homeomorphisms of X^* have a local representation. So they're all trivial.

- If $\phi \in \mathrm{Homeo}(X^*)$, let $\tilde{\phi}$ the dual automorphism of $C(X^*)$.

Theorem (V.,'17)

Assume Forcing Axioms and suppose that X is connected and second-countable. Then all homeomorphisms of X^* have a local representation. So they're all trivial.

- If $\phi \in \mathrm{Homeo}(X^*)$, let $\tilde{\phi}$ the dual automorphism of $C(X^*)$.
- For $a \in D(X)$, say $a = \{a_n\}$, we find open sets U_n with disjoint closure such that $a_n \in U_n$. Note that $\prod C_0(U_n)/\bigoplus C_0(U_n) \subseteq C(X^*)$.

Theorem (V.,'17)

Assume Forcing Axioms and suppose that X is connected and second-countable. Then all homeomorphisms of X^* have a local representation. So they're all trivial.

- If $\phi \in \mathrm{Homeo}(X^*)$, let $\tilde{\phi}$ the dual automorphism of $C(X^*)$.
- For $a \in D(X)$, say $a = \{a_n\}$, we find open sets U_n with disjoint closure such that $a_n \in U_n$. Note that $\prod C_0(U_n)/\bigoplus C_0(U_n) \subseteq C(X^*)$.
- The goal is to produce a nice lifting of $\tilde{\phi} \restriction \prod C_0(U_n)/\bigoplus C_0(U_n)$

Theorem (V.,'17)

Assume Forcing Axioms and suppose that X is connected and second-countable. Then all homeomorphisms of X^* have a local representation. So they're all trivial.

- If $\phi \in \mathrm{Homeo}(X^*)$, let $\tilde{\phi}$ the dual automorphism of $C(X^*)$.
- For $a \in D(X)$, say $a = \{a_n\}$, we find open sets U_n with disjoint closure such that $a_n \in U_n$. Note that $\prod C_0(U_n)/\bigoplus C_0(U_n) \subseteq C(X^*)$.
- The goal is to produce a nice lifting of $\tilde{\phi} \restriction \prod C_0(U_n)/\bigoplus C_0(U_n)$
- We use the finite-dimensional approximation property of $C(X^*)$ to filter $\prod C_0(U_n)/\bigoplus C_0(U_n)$ into a net, indexed by $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, of spaces of the form $\prod E_n/\bigoplus E_n$ with E_n finite-dimensional Banach spaces.

Theorem (V.,'17)

Assume Forcing Axioms and suppose that X is connected and second-countable. Then all homeomorphisms of X^* have a local representation. So they're all trivial.

- If $\phi \in \mathrm{Homeo}(X^*)$, let $\tilde{\phi}$ the dual automorphism of $C(X^*)$.
- For $a \in D(X)$, say $a = \{a_n\}$, we find open sets U_n with disjoint closure such that $a_n \in U_n$. Note that $\prod C_0(U_n)/\bigoplus C_0(U_n) \subseteq C(X^*)$.
- The goal is to produce a nice lifting of $\tilde{\phi} \restriction \prod C_0(U_n)/\bigoplus C_0(U_n)$
- We use the finite-dimensional approximation property of $C(X^*)$ to filter $\prod C_0(U_n)/\bigoplus C_0(U_n)$ into a net, indexed by $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, of spaces of the form $\prod E_n/\bigoplus E_n$ with E_n finite-dimensional Banach spaces.
- We then apply a very technical lifting theorem of McKenney and V. for well behaved maps whose domain is of the form $\prod E_n/\bigoplus E_n \to C(X^*)$, where E_n are finite-dimensional Banach spaces.

Theorem (V.,'17)

Assume Forcing Axioms and suppose that X is connected and second-countable. Then all homeomorphisms of X^* have a local representation. So they're all trivial.

- If $\phi \in \mathrm{Homeo}(X^*)$, let $\tilde{\phi}$ the dual automorphism of $C(X^*)$.
- For $a \in D(X)$, say $a = \{a_n\}$, we find open sets U_n with disjoint closure such that $a_n \in U_n$. Note that $\prod C_0(U_n)/\bigoplus C_0(U_n) \subseteq C(X^*)$.
- The goal is to produce a nice lifting of $\tilde{\phi} \restriction \prod C_0(U_n)/\bigoplus C_0(U_n)$
- We use the finite-dimensional approximation property of $C(X^*)$ to filter $\prod C_0(U_n)/\bigoplus C_0(U_n)$ into a net, indexed by $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, of spaces of the form $\prod E_n/\bigoplus E_n$ with E_n finite-dimensional Banach spaces.
- We then apply a very technical lifting theorem of McKenney and V. for well behaved maps whose domain is of the form $\prod E_n/\bigoplus E_n \to C(X^*)$, where E_n are finite-dimensional Banach spaces. We produce "nice" liftings $\prod E_n \to C_b(X)$, again indexed by $(\mathbb{N}^{\mathbb{N}}, \leq^*)$.

Theorem (V.,'17)

Assume Forcing Axioms and suppose that X is connected and second-countable. Then all homeomorphisms of X^* have a local representation. So they're all trivial.

- If $\phi \in \text{Homeo}(X^*)$, let $\tilde{\phi}$ the dual automorphism of $C(X^*)$.
- For $a \in D(X)$, say $a = \{a_n\}$, we find open sets U_n with disjoint closure such that $a_n \in U_n$. Note that $\prod C_0(U_n)/\bigoplus C_0(U_n) \subseteq C(X^*)$.
- The goal is to produce a nice lifting of $\tilde{\phi} \restriction \prod C_0(U_n)/\bigoplus C_0(U_n)$
- We use the finite-dimensional approximation property of $C(X^*)$ to filter $\prod C_0(U_n)/\bigoplus C_0(U_n)$ into a net, indexed by $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, of spaces of the form $\prod E_n/\bigoplus E_n$ with E_n finite-dimensional Banach spaces.
- We then apply a very technical lifting theorem of McKenney and V. for well behaved maps whose domain is of the form $\prod E_n/\bigoplus E_n \to C(X^*)$, where E_n are finite-dimensional Banach spaces. We produce "nice" liftings $\prod E_n \to C_b(X)$, again indexed by $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. We then use stronger versions of OCA to uniformize these liftings, and then some perturbation theory for C^*-algebras, to obtain *-homomorphisms $\prod C_0(U_n) \to C_b(X)$ lifting $\tilde{\phi}$;
- We then use duality to go back to a topological setting and find b such that $\phi[a^*] = b^*$.

Theorem (V., '17)

Let X be connected and second-countable and assume Forcing Axioms. Then all homeomorphisms of X^* are trivial.

Theorem (V., '17)

Let X be connected and second-countable and assume Forcing Axioms. Then all homeomorphisms of X^* are trivial.

	$CH \Rightarrow \exists$ nontrivial	Forcing Axioms $\Rightarrow$ all trivial
$X = \mathbb{N}$	Rudin	Velickovic
$dim(X) = 0$	Parovicenko	Farah, Farah-McKenney
$X = [0, 1)$, $X = \mathbb{R}$	Yu (but see K.P. Hart)	V.
$X = \bigsqcup X_i$, X_i cpct	Coskey-Farah, Farah-Shelah	
X manifold	V.	V.
X connected		V.

PFA: Disconnected spaces

What if X is non connected?

PFA: Disconnected spaces

What if X is non connected? If $X = \sqcup X_i$, where each X_i is compact and connected, similar techniques to the ones of Farah-McKenney, again making heavy use of the lifting theorem of McKenney-V. and of some perturbation theory lead to the following

PFA: Disconnected spaces

What if X is non connected? If $X = \sqcup X_i$, where each X_i is compact and connected, similar techniques to the ones of Farah-McKenney, again making heavy use of the lifting theorem of McKenney-V. and of some perturbation theory lead to the following:

Theorem (McKenney-V.)

Let $X = \sqcup X_i$ where X_i is compact, second-countable and connected. Assume Forcing Axioms. Then

- *all homeomorphisms of X^* are trivial*

What if X is non connected? If $X = \sqcup X_i$, where each X_i is compact and connected, similar techniques to the ones of Farah-McKenney, again making heavy use of the lifting theorem of McKenney-V. and of some perturbation theory lead to the following:

Theorem (McKenney-V.)

Let $X = \sqcup X_i$ where X_i is compact, second-countable and connected. Assume Forcing Axioms. Then

- *all homeomorphisms of X^* are trivial*
- *If $Y = \sqcup Y_i$, where each Y_i is compact, second-countable and connected, then X^* and Y^* are homeomorphic if and only if there is a g, an almost perturbation of $\mathbb{N}$, such that $\forall^\infty n$ X_n is homeomorphic to $Y_{g(n)}$.*

PFA: Disconnected spaces

What if X is non connected? If $X = \sqcup X_i$, where each X_i is compact and connected, similar techniques to the ones of Farah-McKenney, again making heavy use of the lifting theorem of McKenney-V. and of some perturbation theory lead to the following:

Theorem (McKenney-V.)

Let $X = \sqcup X_i$ where X_i is compact, second-countable and connected. Assume Forcing Axioms. Then

- *all homeomorphisms of X^* are trivial*
- *If $Y = \sqcup Y_i$, where each Y_i is compact, second-countable and connected, then X^* and Y^* are homeomorphic if and only if there is a g, an almost perturbation of $\mathbb{N}$, such that $\forall^\infty n$ X_n is homeomorphic to $Y_{g(n)}$.*

Theorem (Farah-Shelah + Ghasemi)

Assume CH. There are connected compact spaces X_n and Y_n such that no X_n is homeomorphic to Y_n but, if $X = \sqcup X_n$ and $Y = \sqcup Y_n$, X^* and Y^* are homeomorphic.

What if X is non connected? If $X = \sqcup X_i$, where each X_i is compact and connected, similar techniques to the ones of Farah-McKenney, again making heavy use of the lifting theorem of McKenney-V. and of some perturbation theory lead to the following:

Theorem (McKenney-V.)

Let $X = \sqcup X_i$ where X_i is compact, second-countable and connected. Assume Forcing Axioms. Then

- *all homeomorphisms of X^* are trivial*
- *If $Y = \sqcup Y_i$, where each Y_i is compact, second-countable and connected, then X^* and Y^* are homeomorphic if and only if there is a g, an almost perturbation of $\mathbb{N}$, such that $\forall^\infty n$ X_n is homeomorphic to $Y_{g(n)}$.*

Theorem (Farah-Shelah + Ghasemi)

Assume CH. There are connected compact spaces X_n and Y_n such that no X_n is homeomorphic to Y_n but, if $X = \sqcup X_n$ and $Y = \sqcup Y_n$, X^* and Y^* are homeomorphic.

If X is disconnected but not of the form $X = \sqcup X_i$: work in progress.

	$CH \Rightarrow \exists$ nontrivial	Forcing Axioms $\Rightarrow$ all trivial
$X = \mathbb{N}$	Rudin	Velickovic
$dim(X) = 0$	Parovicenko	Farah, Farah-McKenney
$X = [0,1)$, $X = \mathbb{R}$	Yu (but see K.P. Hart)	V.
$X = \bigsqcup X_i$, X_i cpct	Coskey-Farah, Farah-Shelah	McKenney-V.
X manifold	V.	V.
X connected		V.

Theorem (V., '17)

Assume Forcing Axioms and let X, Y be connected and second-countable. Then X^* is homeomorphic to Y^* only if there are compact $K_1 \subseteq X$ and $K_2 \subseteq Y$ such that $X \setminus K_1$ is homeomorphic to $Y \setminus K_2$.

Theorem (V., '17)

Assume Forcing Axioms and let X, Y be connected and second-countable. Then X^* is homeomorphic to Y^* only if there are compact $K_1 \subseteq X$ and $K_2 \subseteq Y$ such that $X \setminus K_1$ is homeomorphic to $Y \setminus K_2$.

Question

Is there a second-countable Y such that under CH Y^* is homeomorphic to $(\mathbb{R}^n)^*$, for some $n \in \mathbb{N}$, but under Forcing Axioms this is not the case?

Thank you!