Some games and their topological consequences III

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ICMC-USP (Partially supported by FAPESP)
Baire spaces

Definition

A topological space is Baire if for every family $\{A_n : n \in \omega\}$ of open dense subsets, $\bigcap_{n \in \omega} A_n$ is dense.
Baire spaces

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A classical game

The Banach-Mazur game is played as follows:

• Alice plays $A_0$, a non-empty open set;
• Bob plays $B_0 \subset A_0$, a non-empty open set;
• Alice plays $A_1 \subset B_0$, a non-empty open set;
• Bob plays $B_1 \subset A_1$, a non-empty open set;
• and so on, for every $n \in \omega$.

At the end, Bob is declared the winner if $\bigcap_{n \in \omega} B_n \neq \emptyset$ and Alice is the winner otherwise.
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A classical relation

Theorem (Oxtoby)

$x$ is a Baire space if and only if Alice does not have a winning strategy for the Banach-Mazur game.
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$X$ is a Baire space if and only if Alice does not have a winning strategy for the Banach-Mazur game.
Warming up

Suppose that \( X \) is not Baire. Let us show that Alice has a winning strategy. Let \( V \) be a non-empty open set and let \( \langle A_n : n \in \omega \rangle \) be a sequence of open dense subsets such that \( V \cap \bigcap_{n \in \omega} A_n = \emptyset \).

- Alice plays \( V \cap A_0 \);
- Bob plays \( B_0 \subset (V \cap A_0) \);
- Alice plays \( B_0 \cap A_1 \);
- and so on.

Since \( V \cap \bigcap_{n \in \omega} A_n = \emptyset \), \( \bigcap_{n \in \omega} B_n = \emptyset \).

Poor Bob.
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Suppose that $X$ is not Baire. Let us show that ALICE has a winning strategy. Let $V$ be a non-empty open set and let $\langle A_n : n \in \omega \rangle$ be a sequence of open dense subsets such that $V \cap \bigcap_{n \in \omega} A_n = \emptyset$. 
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Warming up

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- \texttt{Alice} plays $V \cap A_0$;
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- Alice plays $V \cap A_0$;
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- Alice plays $V \cap A_0$;
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- and so on.

Since $V \cap \bigcap_{n \in \omega} A_n = \emptyset$, $\bigcap_{n \in \omega} B_n = \emptyset$. Poor Bob.
The other direction

Now suppose that $X$ is Baire. Let $\sigma$ be a strategy for Alice. We will show that $\sigma$ is not winning. Since $X$ is Baire, so it is $V = \sigma(\langle \rangle)$. Lemma $\bigcup B \in \tau \subset \sigma(\langle B \rangle)$ is open dense in $V$.

Proof. Let $W \subset V$ be a non-empty open set. Then $\emptyset \neq \sigma(\langle W \rangle) \subset W$. Let $S_n = \{\text{all possible Alice's plays at the } n\text{-th inning}\}$. Note that the above lemma just tells us that $\bigcup A \in S_1 A$ is open dense in $V$. And basically with the same proof, $D_n = \bigcup A \in S_n A$ is open dense in $V$ for every $n$. 
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Now suppose that $X$ is Baire. Let $\sigma$ be a strategy for $\text{ALICE}$. We will show that $\sigma$ is not winning.

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**Lemma**

$\bigcup_{B \in \mathcal{T}_CA} \sigma(\langle B \rangle)$ is open dense in $V$. 
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**Lemma**

$\bigcup_{B \in \tau_C} \sigma(\langle B \rangle)$ is open dense in $V$.

**Proof.**

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**Lemma**

$\bigcup_{B \in \tau^A} \sigma(\langle B \rangle)$ is open dense in $V$.

**Proof.**

Let $W \subset V$ be a non-empty open set. Then

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$$\emptyset \neq \sigma(\langle W \rangle) \subset W$$

Let $S_n = \{\text{all possible Alice's plays at the } n\text{-th inning}\}$.

Note that the above lemma just tells us that $\bigcup_{A \in S_1} A$ is open dense in $V$. And basically with the same proof, $D_n = \bigcup_{A \in S_n} A$ is open dense in $V$ for every $n$. 
Bob can find a way

Since \( V \) is Baire, there is an \( x \in \bigcap_{n \in \omega} D_n \).

Now Bob just has to follow this \( x \).

At the inning \( n \), Bob just picks an open set that has \( x \) in its interior.

Since \( x \) is in the intersection, the answer from Alice will also contain \( x \).

We may have a problem here.
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Since $V$ is Baire, there is an $x \in \bigcap_{n \in \omega} D_n$. 

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How to solve it

We have to change a bit the definition of the $D$'s. Instead of just looking for the possible answers, we look for maximal antichains (and one being a refinement of the previous one).
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Instead of just looking for the possible answers, we look for maximal antichains (and one being a refinement of the previous one).
It is better if we draw a picture
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Products

Theorem

There are Baire spaces $X$ and $Y$ such that $X \times Y$ is not Baire.

Let us call a space $X$ productively Baire if $X \times Y$ is Baire for all Baire space $Y$.

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If Bob has a winning strategy for the Banach-Mazur game, then the space is productively Baire.
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Let us call a space $X$ **productively Baire** if $X \times Y$ is Baire for all Baire space $Y$.

**Theorem**

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Let \( b \) be a winning strategy for Bob in the Banach-Mazur game over \( X \). Let \( ea \) be a winning strategy for Alice in the Banach-Mazur game over \( X \times Y \) (since it is not Baire).
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First note that we can suppose that $ea$ always plays basic open sets.

Let $V_0 \times W_0 = ea(\langle \rangle)$. 

\[ b \text{ can take care of it, so let } B_0 = b(\langle V_0 \rangle). \]

Alice plays $W_0$ in the play over $Y$.

Some Bob plays $U_0 \subset W_0$ on $Y$.

We go back to $X \times Y$ and let $V_1 \times W_1 = ea(\langle B_0 \times U_0 \rangle)$.

Start over.

The point is, \( \bigcap_{n \in \mathbb{N}} B_n \) is non-empty.

\[ \bigcap_{n \in \mathbb{N}} B_n \times U_n = \emptyset. \]

So \( \bigcap_{n \in \mathbb{N}} W_n = \emptyset. \)
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Some Bob plays $U_0 \subset W_0$ on $Y$.

We go back to $X \times Y$ and let $V_1 \times W_1 = ea(\langle B_0 \times U_0 \rangle)$.

Start over.
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Start over.

The point is, $\bigcap_{n \in \omega} B_n$ is non-empty.
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Alice plays \( W_0 \) in the play over \( Y \).

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The point is, \( \bigcap_{n \in \omega} B_n \) is non-empty. \( \bigcap_{n \in \omega} B_n \times U_n = \emptyset \).
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First note that we can suppose that \( ea \) always plays basic open sets.

Let \( V_0 \times W_0 = ea(\langle \rangle) \).

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\textsc{Alice} plays \( W_0 \) in the play over \( Y \).

Some \textsc{Bob} plays \( U_0 \subset W_0 \) on \( Y \).

We go back to \( X \times Y \) and let \( V_1 \times W_1 = ea(\langle B_0 \times U_0 \rangle) \).

Start over.

The point is, \( \bigcap_{n \in \omega} B_n \) is non-empty. \( \bigcap_{n \in \omega} B_n \times U_n = \emptyset \). So \( \bigcap_{n \in \omega} W_n = \emptyset \).
It is not so confusing
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Proposition

Let $X \subset \mathbb{R}$. If Bob has a winning strategy for the Banach-Mazur game over $X$, then $X$ has a Cantor subspace.
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Parallel realities
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Bernstein

Definition
We say that $X \subseteq \mathbb{R}$ is a Bernstein set if it is uncountable and, for every uncountable closed set $F \subseteq X$, $F \cap X \neq \emptyset$ and $F \cap (\mathbb{R} \setminus X)$ are both non-empty.

Corollary
If $X$ is a Bernstein set, then Bob has no winning strategy.
Definition
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Definition
We say that $X \subseteq \mathbb{R}$ is a **Bernstein set** if it is uncountable and, for every uncountable closed set $F \subseteq X$, $F \cap X$ e $F \cap (\mathbb{R} \setminus X)$ are both non-empty.

Corollary
*If $X$ is a Bernstein set, then Bob has no winning strategy.*
Changing the game a little bit

Let us make Bob’s life easier:

• Alice plays $A_0$, a non-empty open set;
• Bob plays $B_{10}$, $B_{20} \subset A_0$, non-empty open sets;

Let us define $B_0 = B_{10} \cup B_{20}$;

• Alice plays $A_{11} \subset B_{10}$, $A_{21} \subset B_{20}$ non-empty open sets;

• Bob plays $B_{11}$, $B_{21} \subset A_{11}$ and $B_{31}$, $B_{41} \subset A_{21}$ non-empty open sets;

Let us define $B_{11} = B_{11} \cup B_{21} \cup B_{31} \cup B_{41}$.

• And so on.

Bob is declared the winner if $\bigcap_{n \in \omega} B_n \neq \emptyset$ and Alice is the winner otherwise.
Changing the game a little bit

Let us make Bob’s life easier:

• Alice plays $A_0$ non-empty open set;
• Bob plays $B_1^0, B_2^0 \subset A_0$ non-empty open sets;

Let us define $B_0^0 = B_1^0 \cup B_2^0$;

• Alice plays $A_1^1, A_2^1 \subset B_1^0$, $A_2^1 \subset B_2^0$ non-empty open sets;
• Bob plays $B_1^1, B_2^1 \subset A_1^1$ and $B_3^1, B_4^1 \subset A_2^1$ non-empty open sets;

Let us define $B_1^1 = B_1^1 \cup B_2^1 \cup B_3^1 \cup B_4^1$.

• And so on.

Bob is declared the winner if $\bigcap_{n \in \omega} B_n \neq \emptyset$ and Alice is the winner otherwise.
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Let us make Bob’s life easier:

- Alice plays $A_0$ non-empty open set;

And so on. Bob is declared the winner if $\bigcap_{n \in \mathbb{N}} B_n \neq \emptyset$ and Alice is the winner otherwise.
Changing the game a little bit

Let us make Bob's life easier:

- **Alice** plays $A_0$ non-empty open set;
- **Bob** plays $B_0^1, B_0^2 \subset A_0$ non-empty open sets;
Changing the game a little bit

Let us make Bob’s life easier:

- Alice plays $A_0$ non-empty open set;
- Bob plays $B_1^0, B_2^0 \subset A_0$ non-empty open sets; Let us define $B_0 = B_1^0 \cup B_2^0$;
Changing the game a little bit

Let us make Bob’s life easier:

- Alice plays $A_0$ non-empty open set;
- Bob plays $B_0^1, B_0^2 \subset A_0$ non-empty open sets; Let us define $B_0 = B_0^1 \cup B_0^2$;
- Alice plays $A_1^1 \subset B_0^1, A_1^2 \subset B_0^2$ non-empty open sets;
Changing the game a little bit

Let us make Bob’s life easier:

- **Alice** plays $A_0$ non-empty open set;
- **Bob** plays $B_{01}, B_{02} \subset A_0$ non-empty open sets; Let us define $B_0 = B_{01} \cup B_{02}$;
- **Alice** plays $A_{11} \subset B_{01}, A_{12} \subset B_{02}$ non-empty open sets;
- **Bob** plays $B_{11}, B_{12} \subset A_{11}$ and $B_{13}, B_{14} \subset A_{12}$ non-empty open sets;
Changing the game a little bit

Let us make Bob’s life easier:

- **Alice** plays $A_0$ non-empty open set;
- **Bob** plays $B_0^1, B_0^2 \subset A_0$ non-empty open sets; Let us define $B_0 = B_0^1 \cup B_0^2$;
- **Alice** plays $A_1^1 \subset B_0^1$, $A_1^2 \subset B_0^2$ non-empty open sets;
- **Bob** plays $B_1^1, B_1^2 \subset A_1^1$ and $B_1^3, B_1^4 \subset A_1^2$ non-empty open sets; Let us define $B_1 = B_1^1 \cup B_1^2 \cup B_1^3 \cup B_1^4$. 

And so on.

Bob is declared the winner if $\cap_{n \in \omega} B_n \neq \emptyset$ and Alice is the winner otherwise.
Changing the game a little bit

Let us make Bob’s life easier:

- Alice plays $A_0$ non-empty open set;
- Bob plays $B_1^1, B_2^2 \subseteq A_0$ non-empty open sets; Let us define $B_0 = B_1^1 \cup B_2^2$;
- Alice plays $A_1^1 \subseteq B_1^1, A_1^2 \subseteq B_2^2$ non-empty open sets;
- Bob plays $B_1^3, B_1^4 \subseteq A_1^1$ and $B_3^3, B_4^4 \subseteq A_1^2$ non-empty open sets; Let us define $B_1 = B_1^1 \cup B_2^2 \cup B_3^3 \cup B_4^4$.
- And so on.
Changing the game a little bit

Let us make Bob’s life easier:

- Alice plays $A_0$ non-empty open set;
- Bob plays $B_0^1, B_0^2 \subset A_0$ non-empty open sets; Let us define $B_0 = B_0^1 \cup B_0^2$;
- Alice plays $A_1^1 \subset B_0^1, A_1^2 \subset B_0^2$ non-empty open sets;
- Bob plays $B_1^1, B_1^2 \subset A_1^1$ and $B_1^3, B_1^4 \subset A_1^2$ non-empty open sets; Let us define $B_1 = B_1^1 \cup B_1^2 \cup B_1^3 \cup B_1^4$.
- And so on.

Bob is declared the winner if $\bigcap_{n \in \omega} B_n \neq \emptyset$ and Alice is the winner otherwise.
Products again

Like we did before, it is possible to show (about this new game) that:

• Alice has a winning strategy if, and only if, the space is not Baire;
• If Bob has a winning strategy, then the space is productively Baire.

Are these games different?
Like we did before, it is possible to show (about this new game) that

- Alice has a winning strategy if, and only if, the space is not Baire;
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Bernstein again

Proposition

If $X$ is a Bernstein set, then Bob has a winning strategy for this new game.
Bernstein again

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Think that you are playing over 🎲
Think that you are playing over R
Think that you are playing over $R$
Think that you are playing over R
Multiboard game

Let $BM_2$ be the 2-boards game version of Banach-Mazur. There are two boards of the game, Alice starts playing on all the boards. Then Bob answers playing in all the boards (following the rules on each board). Then Alice again and so on. We say that Bob wins the game if he wins on all boards. Alice is the winner otherwise (i.e., Alice wins at some board).
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Let $BM^\kappa$ be the $\kappa$-boards game version of Banach-Mazur. There are $\kappa$ boards of the game, Alice starts playing on all the boards. Then Bob answers playing in all the boards (following the rules on each board). Then Alice again and so on.

We say that Bob wins the game if he wins on all boards. Alice is the winner otherwise (i.e. Alice wins at some board).
Seeing the games
Seeing the games
Seeing the games
So...

• If Bob has a winning strategy for $BM_1$, he has one for $BM_\kappa$.

• If Alice has a winning strategy for $BM_1$, she has one for $BM_\kappa$.

• If Bob has a winning strategy for $BM_\kappa$, he has one for $BM_1$.

• If you start with a Baire space where Bob does not have a winning strategy for $BM_1$ and $BM_\kappa$ is determined, then Alice has a winning strategy for the $BM_\kappa$.

• Given a space $X$, can we always find a $\kappa$ such as $BM_\kappa$ is determined?

• Yes, kind of.

• If it is consistent that there is a proper class of measurable cardinals, then the above conjecture is consistently true. [1]

• The motivation for the conjecture was: if Bob has a winning strategy for $BM_1$ on $X$, then $\Box_\xi<\kappa X$ is Baire for any $\kappa$.

Is the converse also true?
So...

- If Bob has a winning strategy for $BM^1$, he has one for $BM^\kappa$.

- If Alice has a winning strategy for $BM^1$, she has one for $BM^\kappa$.

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F. Galvin and M. Scheepers.

**Baire spaces and infinite games.**