Fraïssé-like structures

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Introduction

Let $\mathcal{L} = \{R_1, R_2, ...\}$ be a first order relational language and X a countable structure in $\mathcal{L}$.

Definition

We say that that X is ultrahomogeneous if for every finite substructures $A, B \leq X$ and every isomorphism $\varphi : A \to B$ there is an automorphism $\phi : X \to X$ such that $\phi \restriction A = \varphi$.

Definition

$Age(X) := \{A \subseteq X : |A| < \omega\}$

Proposition

If X is ultrahomogeneous then $\mathcal{C} := Age(X)$

- is countable up to isomorphism,
- has the hereditary property (HP),
- has the joint embedding property (JEP),
- has the amalgamation property (AP).

Definition

We say that a countable class of finite structures $\mathcal{C}$ that satisfies HP,JEP and AP is a Fraïssé class.

Theorem(Fraïssé)

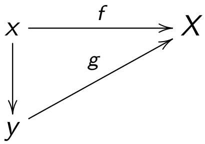
Let $\mathcal{C}$ be a Fraïssé class. Then there is an ultrahomogeneous structure X, called the Fraïssé limit of $\mathcal{C}$ (Flim($\mathcal{C}$)), such that $Age(X) = \mathcal{C}$.

Examples

- $\mathcal{C}$ =finite linear orders, Flim($\mathcal{C}$) $= \mathbb{Q}$,
- $\mathcal{C}$ =finite graphs, Flim($\mathcal{C}$) =Random graph,
- $\mathcal{C}$ =finite rational metric spaces, Flim($\mathcal{C}$) =rational Urysohn space,
- ...

Definition

We say that a structure X satisfies the extension property with respect to $\mathcal{C} \subseteq Age(X)$ if for every $x \leq y \in \mathcal{C}$ and every $f : x \to X$ there is $g : y \to X$ such that the following diagram



commutes.

Proposition

Let $\mathcal{C}$ be a Fraïssé class and X countable structure such that $Age(X) = \mathcal{C}$. Then $X = \mathrm{Flim}(\mathcal{C})$ iff X satisfies the extension property with respect to $\mathcal{C}$.

Definition

Let $\mathcal{C}$ be a Fraïssé class. We say that a structure X is Fraïssé-like if $Age(X) = \mathcal{C}$ and it satisfies the extension property with respect to $\mathcal{C}$.

We are particularly interested in the case when X is a Fraïssé-like structure of cardinality ω_1. In that case there are embeddings $\{e_i\}_{i<\omega_1}$ such that X is a colimit of a chain

$$\mathrm{Flim}(\mathcal{C}) \xrightarrow{e_0} \mathrm{Flim}(\mathcal{C}) \xrightarrow{e_1} \mathrm{Flim}(\mathcal{C}) \xrightarrow{e_2} \ldots$$

Questions

Are Fraïssé-like structures of cardinality ω_1 uniquely determined? Are they ultrahomogeneous? What can we say about their automorphism groups?

To understand how can automorphism groups of Fraïssé-like structures look like we must understand the following.

Definition

To every $e : \mathsf{Flim}(\mathcal{C}) \to \mathsf{Flim}(\mathcal{C})$ we assign $G_e \leq \mathsf{Aut}(\mathsf{Flim}(\mathcal{C}))$ such that $\alpha \in \mathsf{Aut}(\mathsf{Flim}(\mathcal{C}))$ is in G_e iff there is $\beta \in \mathsf{Aut}(\mathsf{Flim}(\mathcal{C}))$ such that the following diagram commutes

$$\begin{array}{ccc} \mathsf{Flim}(\mathcal{C}) & \xrightarrow{e} & \mathsf{Flim}(\mathcal{C}) \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\beta} \\ \mathsf{Flim}(\mathcal{C}) & \xrightarrow[e]{} & \mathsf{Flim}(\mathcal{C}) \end{array}.$$

Let $\mathcal{G}$ be the class of finite graphs. The Fraïssé limit is the Random graph $\mathcal{R}$.

Theorem(Imrich-Klavžar-Trofimov)

There is $e : \mathcal{R} \to \mathcal{R}$ such that $|G_e| = 1$.

Theorem

There exists Fraïssé-like graphs X_0, X_1 of cardinality ω_1 such that X_0 is ultrahomogeneous and X_1 is rigid.

The same can be proved for the class of finite metric spaces where the Fraïssé limit is the Urysohn space $\mathbb{U}$.

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- W. Imrich, S. Klavžar, V. Trofimov, *Distinguishing infinite graphs*, Electron. J. Combin. 14 (2007), no. 1, Research Paper 36, 12 pp.