One some Pradoxical Point Sets in Different Models of Set Theory

Mariam Beriashvili

[1] I. Vekua Institute of Applied Mathematics
Tbilisi State Univeristy

[2] Georgian Technical University

February 2, 2017

Paradoxical Subsets of **R** in **ZFC**

- **Vitali Set, 1905**
  Any selector of the quotient-set $\mathbf{R}/\mathbf{Q}$ (where $\mathbf{Q}$ stands for the field of all rational numbers) is a Vitali Set.

- **Vitali Set, 1905**
  Any selector of the quotient-set **R**/**Q** (where **Q** stands for the field of all rational numbers) is a Vitali Set.
- **Hamel Bases, 1905**
  Any bases for the real line **R** considered as a vector space over the field **Q** is a Hamel Bases.

Paradoxical Subsets of $\mathbf{R}$ in $\mathbf{ZFC}$

- $\mathbf{Vitali\quad Set, 1905}$
  Any selector of the quotient-set $\mathbf{R}/\mathbf{Q}$ (where $\mathbf{Q}$ stands for the field of all rational numbers) is a Vitali Set.
- $\mathbf{Hamel\quad Bases, 1905}$
  Any bases for the real line $\mathbf{R}$ considered as a vector space over the field $\mathbf{Q}$ is a Hamel Bases.
- $\mathbf{Bernstein\quad Set, 1908}$ We say that $X \subset \mathbf{R}$ is a Bernstein set if, for every non-empty perfect set $P \subset \mathbf{R}$, both intersections

  $$P \cap X \text{ and } P \cap (\mathbf{R} \setminus X)$$

  are nonempty.

Paradoxical Subsets of $\mathbf{R}$ in $\mathbf{ZFC} + \mathbf{CH}$

- **Luzini Set, 1914**
  A set $X \subset \mathbb{R}$ is a Luzin set if X is uncountable and, for every first category set $Y \subset \mathbb{R}$, the inequality $\mathrm{card}(X \cap Y) \leq \omega$ holds true.

- **Luzini Set, 1914**
  A set $X \subset \mathbb{R}$ is a Luzin set if X is uncountable and, for every first category set $Y \subset \mathbb{R}$, the inequality $\mathrm{card}(X \cap Y) \leq \omega$ holds true.
- **Sierpiński Set, 1924**
  A set $X \subset \mathbb{R}$ is called a Sierpiński set if X is uncountable and, for every λ-measure zero set $Y \subset \mathbb{R}$, the inequality $\mathrm{card}(X \cap Y) \leq \omega$ holds true.

Relationships between Paradoxical Sets

Theorem

- *There exists a subset X of $\mathbf{R}$ such that X is simultaneously a Vitali set and a Bernstein set.*

Relationships between Paradoxical Sets

Theorem

- *There exists a subset X of $\mathbf{R}$ such that X is simultaneously a Vitali set and a Bernstein set.*
- *There exists a Hamel basis of $\mathbf{R}$ which simultaneously is a Bernstein set.*

Theorem

- *There exists a subset X of $\mathbf{R}$ such that X is simultaneously a Vitali set and a Bernstein set.*
- *There exists a Hamel basis of $\mathbf{R}$ which simultaneously is a Bernstein set.*
- *There exists no Hamel basis in $\mathbf{R}$ which simultaneously is a Vitali set.*

Relationships between Paradoxical Sets

Theorem

- *There exists no a subset X of $\mathbf{R}$ such that X is simultaneously a Luzini set (Sierpiński Set) and a Bernstein set.*

Relationships between Paradoxical Sets

Theorem

- *There exists no a subset X of* $\mathbf{R}$ *such that X is simultaneously a Luzini set (Sierpiński Set) and a Bernstein set.*
- *There exists a Hamel basis of* $\mathbf{R}$ *which simultaneously is a Luzini set (Sierpiński Set).*

Relationships between Paradoxical Sets

Theorem

- *There exists no a subset X of* $\mathbf{R}$ *such that X is simultaneously a Luzini set (Sierpiński Set) and a Bernstein set.*
- *There exists a Hamel basis of* $\mathbf{R}$ *which simultaneously is a Luzini set (Sierpiński Set).*
- *There exists no Vitali set* $\mathbf{R}$ *which simultaneously is a Luzini set (Sierpiński Set).*

Some Auxiliary Notions

Let E be a set and let $\mathcal{M}$ be a class of measures on E (we assume, in general, that the domains of measures from $\mathcal{M}$ are various σ-algebras of subsets of E).

Definition

- We shall say that a set $X \subset E$ is **absolutely measurable** with respect to $\mathcal{M}$ if, for an arbitrary measure $\mu \in \mathcal{M}$, the set X is measurable with respect to μ.

Some Auxiliary Notions

Let E be a set and let $\mathcal{M}$ be a class of measures on E (we assume, in general, that the domains of measures from $\mathcal{M}$ are various σ-algebras of subsets of E).

Definition

- We shall say that a set $X \subset E$ is **absolutely measurable** with respect to $\mathcal{M}$ if, for an arbitrary measure $\mu \in \mathcal{M}$, the set X is measurable with respect to μ.
- We shall say that a set $Y \subset E$ is **relatively measurable** with respect to the class $\mathcal{M}$ if there exists at least one measure $\mu \in \mathcal{M}$ such that Y is measurable with respect to μ.

Some Auxiliary Notions

Let E be a set and let $\mathcal{M}$ be a class of measures on E (we assume, in general, that the domains of measures from $\mathcal{M}$ are various σ-algebras of subsets of E).

Definition

- We shall say that a set $X \subset E$ is **absolutely measurable** with respect to $\mathcal{M}$ if, for an arbitrary measure $\mu \in \mathcal{M}$, the set X is measurable with respect to μ.
- We shall say that a set $Y \subset E$ is **relatively measurable** with respect to the class $\mathcal{M}$ if there exists at least one measure $\mu \in \mathcal{M}$ such that Y is measurable with respect to μ.
- We shall say that a set $Z \subset E$ is **absolutely nonmeasurable** with respect to $\mathcal{M}$ if there exists no measure $\mu \in \mathcal{M}$ such that Z is measurable with respect to μ.

Marczewski's Method

Let E be a set, μ be a nonzero σ-finite complete measure on some σ-algebra of subsets of E, and let $\mathcal{I}$ be a σ-ideal of subsets of E such that

$$(\forall Y \in \mathcal{I})(\mu_*(Y) = 0),$$

where μ_* stands, as usual, for the inner measure canonically associated with μ. Denote $\mathcal{S} = \mathrm{dom}(\mu)$ and consider the σ-algebra $\mathcal{S}'$ of subsets of E, generated by the union $\mathcal{S} \cup \mathcal{I}$, i.e., $\mathcal{S}' = \sigma(\mathcal{S} \cup \mathcal{I})$. Obviously, any set $Z \in \mathcal{S}'$ can be represented in the form

$$Z = (X \cup Y_1) \setminus Y_2,$$

where $X \in \mathcal{S}$ and both sets Y_1 and Y_2 are some members of $\mathcal{I}$. Then

$$\mu'(Z) = \mu'((X \cup Y_1) \setminus Y_2) = \mu(X).$$

Some Auxiliary Lemmas

Lemma

If a set $Z \subset \mathbb{R}$ is λ-measurable and $\lambda(Z) > 0$, then Z contains a subset Y such that $\mathrm{card}(Y) = \mathbf{c}$ and $\lambda(Y) = 0$.

Lemma

Any member of the σ-ideal generated by the family of all Sierpiński subsets of $\mathbb{R}$ has inner λ-measure zero.

Measurability of Sierpinski Sets

Theorem

There exists a translation invariant measure μ on $\mathbf{R}$ such that:

- μ is an extension of the Lebesgue measure λ;
- all Sierpinski subsets of $\mathbf{R}$ are measurable with respect to μ and all of them have μ-measure zero.

Absolutely non-measurable function

Definition

We say that a function f is absolutely non-measurable with respect to M if there exists no one measure μ such that f is μ -measurable.

Theorem (Kharazishvili)

Let $f : E \to \mathbf{R}$ be a function. The following two assertions are equivalent:

1. f is absolutely nonmeasurable with respect to $M(E)$
2. $ran(f)$ is universal measure zero and $card(f^{-1}(t)) \leq \omega$ for each $t \in \mathbf{R}$.

Theorem(Kharazishvili)

Suppose that there exists a well-ordering $\preceq$ of [0,1] for which the following two conditions are fulfilled

- $\preceq$ is isomorphic to the natural well-ordering of ω_1
- the graph of $\preceq$ is a projecive subset of $[0, 1]^2$.

Then there exists a function

$$\phi : [0, 1] \to [0, 1]$$

whose graph is a projective subset of $[0, 1]^2$ and which is absolutely nonmeasurabel with respect to the class M([0,1]) of all σ-finite diffused nonzero measures on $[0, 1]$

References

1. M. Beriashvili, *Measurable properties of certain paradoxical subsets of the real line,Georgian Mathematical Journal, Vol. 23, Iss. 1, 2016*;
2. M. Beriashvili, *On some paradoxical subsets of the real line, Georgian International Journal of Science and Technology, Volume 6, Number 4, 2014*
3. A. Kharazishvili, *To the existence of projective absolutely nonmeasurable functions, Proc of A. Razmadze Math. Inst., Vol. 166(2014), pp. 95-102*

Thank You for Your Attention!