Homogeneity of ideals

Jacek Tryba

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Joint work with Adam Kwela.
The restriction of the ideal $\mathcal{I}$ to $X \subseteq \bigcup \mathcal{I}$ is given by

$$\mathcal{I}|X = \{A \cap X : A \in \mathcal{I}\}.$$ 

Given two ideals $\mathcal{I}$ and $\mathcal{J}$ we write $\mathcal{I} \cong \mathcal{J}$ if there is a bijection $f : \bigcup \mathcal{I} \longrightarrow \bigcup \mathcal{J}$ such that $f[C] \in \mathcal{J} \iff C \in \mathcal{I}$. 

**Definition (Homogeneity)**

Let $\mathcal{I}$ be an ideal on $\omega$. Then $H(\mathcal{I}) = \{A \subseteq \omega : \mathcal{I}|A \cong \mathcal{I}\}$ is called the homogeneity family of the ideal $\mathcal{I}$.

**Theorem**

When $A \in H(\mathcal{I})$ and $B \supseteq A$ then $B \in H(\mathcal{I})$. 

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*When $A \in H(\mathcal{I})$ and $B \supseteq A$ then $B \in H(\mathcal{I})$.***
We call an ideal $\mathcal{I}$ on $\omega$ *homogeneous* if
\[ H(\mathcal{I}) = \mathcal{I}^+ = \{ A \subseteq X : A \notin \mathcal{I} \}. \]

We call an ideal $\mathcal{I}$ on $\omega$ *anti-homogeneous* if
\[ H(\mathcal{I}) = \mathcal{I}^* = \{ A \subseteq X : A^c \in \mathcal{I} \}. \]
Examples

- Fin, the ideal of all finite sets, is homogeneous.
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- Let $\{ I_n : n \in \omega \}$ be a family of consecutive intervals such that each $I_n$ has length $n!$. An ideal $\mathcal{I} = \{ A \subseteq \omega : \lim_{n \to \infty} |A \cap I_n|/n! = 0 \}$ is anti-homogeneous.
Examples

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- Ideal of sets of asymptotic density zero $\mathcal{I}_d = \{ A \subseteq \mathbb{N} : \lim_{n \to \infty} \frac{|A \cap \{0,1,\ldots,n\}|}{n+1} = 0 \}$ is neither homogeneous nor anti-homogeneous.
Invariant functions

Definition (Balcerzak, Głąb, Swaczyna)

Let $\mathcal{I}$ be an ideal on $\omega$ and $f : \omega \to \omega$ be an injection. We say that $f$ is:

- $\mathcal{I}$-invariant if $f[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$;
- bi-$\mathcal{I}$-invariant if $f[A] \in \mathcal{I} \iff A \in \mathcal{I}$ for all $A \subseteq \omega$.

If $f : \omega \to \omega$ is bi-$\mathcal{I}$-invariant then $f[\omega] \in H(\mathcal{I})$. On the other hand, if $A \in H(\mathcal{I})$ then there is a bi-$\mathcal{I}$-invariant $f : \omega \to \omega$ with $f[\omega] = A$. 

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**Theorem**

The following are equivalent for any ideal $\mathcal{I}$ on $\omega$:
- there is an $\mathcal{I}$-invariant injection $f : \omega \to \omega$ with $\text{Fix}(f) \notin \mathcal{I}^*$ and $f[\omega] \notin \mathcal{I}$;
- $\mathcal{I}$ is not a maximal ideal.

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The following are equivalent for any ideal $\mathcal{I}$ on $\omega$:

- there is a bi-$\mathcal{I}$-invariant injection $f : \omega \rightarrow \omega$ with $\text{Fix}(f) \notin \mathcal{I}^*$;
- there are $A, B \subseteq \omega$ such that $A \Delta B \notin \mathcal{I}$ and $\mathcal{I}\vert A \cong \mathcal{I}\vert B$.

Problem

Characterize the ideals for which there are no $A, B \subseteq \omega$ such that $A \Delta B \notin \mathcal{I}$ and $\mathcal{I}\vert A \cong \mathcal{I}\vert B$. Specifically, find a “nice” example of such an ideal.
Let $\mathcal{I}$ be an ideal on $\omega$. We say that a real sequence $(x_n)_{n \in \omega}$ is $\mathcal{I}$-convergent to $x \in \mathbb{R}$ if for every $\varepsilon > 0$ we have
\[ \{ n \in \omega : |x_n - x| > \varepsilon \} \in \mathcal{I}. \]
Ideal convergence

Let $\mathcal{I}$ be an ideal on $\omega$. We say that a real sequence $(x_n)_{n \in \omega}$ is $\mathcal{I}$-convergent to $x \in \mathbb{R}$ if for every $\varepsilon > 0$ we have

$$\{ n \in \omega : |x_n - x| > \varepsilon \} \in \mathcal{I}.$$ 

**Proposition**

The following are equivalent for any ideal $\mathcal{I}$ on $\omega$ not isomorphic to $\text{Fin} \oplus \mathcal{P}(\omega)$:

- for any sequence $(x_n)_{n \in \omega}$ of reals, $\mathcal{I}$-convergence of $(x_n)_{n \in \omega}$ to some $x \in \mathbb{R}$ implies convergence of $(x_{f(n)})_{n \in \omega}$ to $x$ for some bi-$\mathcal{I}$-invariant injection $f$;

- for every countable family $\{A_n : n \in \omega\} \subseteq \mathcal{I}$ there exists such $A \in H(\mathcal{I})$ that $A \cap A_n$ is finite for every $n \in \omega$.

A homogeneous ideal satisfies the above if and only if it is a weak P-ideal. Moreover, an anti-homogeneous ideal satisfies the above if and only if it is a P-ideal.
Theorem

Let $A \in H(I_d)$ and $\{a_0, a_1 \ldots\}$ be an increasing enumeration of $A$. Then the function $f: \omega \rightarrow A$ given by $f(n) = a_n$ witnesses that $I_d|A \cong I_d$. 
Theorem

Let $A \in H(\mathcal{I}_d)$ and $\{a_0, a_1, \ldots\}$ be an increasing enumeration of $A$. Then the function $f : \omega \to A$ given by $f(n) = a_n$ witnesses that $\mathcal{I}_d|A \cong \mathcal{I}_d$.

Problem

Characterize ideals $\mathcal{I}$ such that for any $A \in H(\mathcal{I})$ the function $f : \omega \to A$ given by $f(n) = a_n$, where $\{a_0, a_1, \ldots\}$ is an increasing enumeration of $A$, witnesses that $\mathcal{I}|A \cong \mathcal{I}$. 