Pseudointersection numbers for Ramsey spaces

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Introduction

Topological Ramsey space theory is an area of Ramsey theory that is concerned with coloring infinite sequences of objects. In the book *Introduction to Ramsey spaces*, Todorčević defines topological Ramsey spaces by extracting properties of the Ellentuck space. Every topological Ramsey space has a notion of pseudointersection number. Dobrinen asked whether the pseudointersection number of each topological Ramsey space is equal to $\mathfrak{p}$, or if not, what the relationships between them are. We find pseudointersection numbers of several topological Ramsey spaces.

The pseudointersection number

Definition

The pseudointersection number $\mathfrak{p}$ is the smallest cardinality of a family of infinite subsets of ω with the strong finite intersection property which does not have a pseudointersection.

Theorem(Bell,1981)

MA implies $\mathfrak{p} = \mathfrak{c}$.

Definition

Let $\mathbb{P}$ be a partial order. We will say that $\mathbb{P}$ is centered if every finite collection of members of $\mathbb{P}$ has a common lower bound. $\mathbb{P}$ is σ-centered if is the union of countably many centered subsets.

Definition

$\mathfrak{m}(\sigma-\text{centered})$ is the smallest cardinal number such that:

1. If $\kappa < \mathfrak{m}(\sigma-\text{centered})$, every collection $\mathcal{D}$ of κ dense subsets of $\mathbb{P}$ admits a $\mathcal{D}$-generic filter.
2. There exists $\mathcal{D}'$ a family of $\mathfrak{m}(\sigma-\text{centered})$ dense subsets of $\mathbb{P}$ which does not admit any $\mathcal{D}$-generic filter.

Theorem (Bell, 1981)

$\mathfrak{m}(\sigma-\text{centered})=\mathfrak{p}$.

We will introduce a topology on $[\omega]^\omega$ called the Ellentuck topology. Let $a \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$ be such that $\max a < \min A$, let

$$[a, A] = \{B \in [\omega]^\omega : a \subseteq B \subseteq a \cup A\}.$$

The *Ellentuck topology* on $[\omega]^\omega$ has as basic open sets the sets of the form $[a, A]$. The space $[\omega]^\omega$ with the Ellentuck topology is called the Ellentuck space.

Definition

A set $X \subseteq [\omega]^\omega$ is called *Ramsey* if for every basic set $[a, A]$ there is $B \in [A]^\omega$ with $[a, B] \subseteq X$ or $[a, B] \cap X = \emptyset$.

Ellentuck's Theorem

Let $X \subseteq [\omega]^\omega$. Then X is Ramsey if and only if X has the Baire Property in the Ellentuck topology.

1. The space $\mathcal{R}_1$ was built by Dobrinen and Todorčević motivated by a Tukey classification problem.
2. $\mathcal{R}_1$ is dense in $\mathcal{ED}^+_{fin}$.
3. $\mathcal{R}_1$ is also dense in a partial order defined by Laflamme which force a weakly Ramsey but not Ramsey ultrafilter.

Definition

Let $X, Y \in \mathcal{R}_1$

1. $X \leq Y$ if $X \subseteq Y$,
2. The $n-th$ finite approximation of X is the set $X \cap (\{l_n\} \times \omega)$ where l_n is the $n-th$ natural number such that $X \cap (\{l_n\} \times \omega) \neq \emptyset$.
3. $X \leq_1^* Y$ if $X \subseteq Y$ modulo an initial segment.

1. The high dimensional Ellentuck space $\mathcal{E}_2$ was built by Natasha Dobrinen to find the Tukey structure of the generic ultrafilter forced by $\mathcal{P}(\omega \times \omega)/\mathrm{Fin} \times \mathrm{Fin}$

Definition

Let $X, Y \in \mathcal{E}_2$

1. $X \leq Y$ if $X \subseteq Y$
2. $X \leq^* Y$ if $X \subseteq Y$ modulo an initial segment of X.

Pseudointersection numbers

Definition

Let $\mathcal{R}$ be a topological Ramsey space. A family $\mathcal{F} \subseteq \mathcal{R}$ has the strong finite intersection property (SFIP) if for every finite subfamily $\{X_1, \ldots, X_n\} \subseteq \mathcal{F}$, there exists $Y \in \mathcal{R}$ such that for each $i \in \{1, \ldots, n\}$, $Y \leq X_i$.

Definition

Given $\mathcal{F} \subseteq \mathcal{R}$, a *pseudointersection* of the family $\mathcal{F}$ is a $Y \in \mathcal{R}$ such that for every $X \in \mathcal{F}$, $Y \leq^* X$.

Pseudointersection number

Definition

Given a topological Ramsey space $\mathcal{R}$, the *pseudointersection number* $\mathfrak{p}_{\mathcal{R}}$ is the smallest cardinality of a family $\mathcal{F} \subseteq \mathcal{R}$ which has the SFIP but does not have a pseudointersection.

Theorem

$\mathfrak{p}_{\mathcal{R}_1} = \mathfrak{p}$

$\mathfrak{p} \leq \mathfrak{p}_{\mathcal{R}_1}$

- Let $\kappa < \mathfrak{p}$ be a cardinal number. Let $\mathcal{F} = \{X_\alpha : \alpha < \kappa\} \subseteq \mathcal{R}_1$ a family with the SFIP.
- The set $\mathbb{P} = \{\langle s, E\rangle : E \in [\kappa]^{<\omega}\}$ is a $\sigma-$centered partial order with the order $\leq$ defined as follows: $\langle s, E\rangle \leq \langle t, F\rangle$ if $t \sqsubseteq s$, $F \subseteq E$ and there exist $X \in \mathcal{R}_1$ such that for each $\alpha \in F$, $X \leq X_\alpha$ and $s \setminus t \subseteq X$.
- For each $\alpha < \kappa$ and $m \in \omega$,

$$D_{\alpha,m} = \{\langle s, E\rangle : \alpha \in E, |s| > m\}$$

is a dense subset of $\mathbb{P}$.

- There exists a filter $\mathcal{G} \subseteq \mathbb{P}$ such that for every $\alpha < \kappa$ and $m \in \omega$, $D_{\alpha,m} \cap \mathcal{G} \neq \emptyset$.
- $Y = \bigcup \{s : \exists E \in [\kappa]^{\omega} : \langle s, E \rangle \in \mathcal{G}\}$ is a pseudointersecton for $\mathcal{F}$
- $\kappa < \mathfrak{p}_{\mathcal{R}_1}$.
- $\mathfrak{p} \leq \mathfrak{p}_{\mathcal{R}_1}$

$\mathfrak{p}_{\mathcal{R}_1} \leq \mathfrak{p}$

- Let $\kappa < \mathfrak{p}_{\mathcal{R}_1}$ be a cardinal number and $\mathcal{F} \subseteq [\omega]^\omega$ a family with the strong finite intersection family.
- There exists a family $\mathcal{F}' \subseteq \mathcal{R}_1$ with the strong finite intersection property such that $|\mathcal{F}'| = |\mathcal{F}|$.

$A = \{a_0, a_1, ..., a_i, ...\} \in \mathcal{F}$ with $a_i < a_{i+1}$

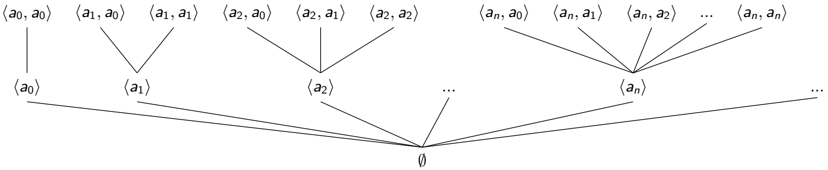

Figure: X_A

- The family $\mathcal{F}'$ admits a pseudointersection.
- $\mathcal{F}$ admits a pseudointersection.
- $\kappa < \mathfrak{p}$.
- $\mathfrak{p}_{\mathcal{R}_1} \leq \mathfrak{p}$.

Hausdorff gaps

Definition

For every $f, g \in \omega^\omega$, we write $f \prec g$ to denote that $\lim_{n \to \infty}(g(n) - f(n)) = \infty$

Definition

A family $\{f_\alpha : \alpha < \omega_1\} \cup \{g_\beta : \beta < \omega_1\} \subseteq \omega^\omega$ is a Hausdorff gap if

- For every $\alpha_1 < \alpha_2 < \omega_1$ and $\beta_1 < \beta_2 < \omega_1$, $f_{\alpha_1} \prec f_{\alpha_2} \prec g_{\beta_2} \prec g_{\beta_1}$.
- For every $\alpha < \omega_1$ and $k < \omega$, $f_\alpha(k) \leq f_\beta(k)$.
- $*$

Hausdorff gaps

Theorem (Hausdorff)

There exists a Hausdorff gap.

- Let $\mathcal{F} \subseteq \{f_\alpha : \alpha < \omega_1\} \cup \{g_\beta : \beta < \omega_1\} \subseteq \omega^\omega$ be a Hausdorff gap.
- Note that if we identify functions with their graphs, $\mathcal{F} \subseteq Fin \otimes Fin$, $\bigcup \mathcal{F} \in (Fin \otimes Fin)^{+}$ and $\mathcal{F}' = \{\omega^2 \setminus f_\alpha : \alpha < \omega_1\} \cup \{\omega^2 \setminus g_\beta : \beta < \omega_1\} \subseteq (Fin \otimes Fin)^{+}$.
- If $\mathcal{F}'$ admits a pseudointersection Y, $Y \cap \bigcup \mathcal{F} \in Fin \otimes Fin$.
- $\bigcup \mathcal{F} \setminus Y \subseteq \omega^2 \setminus Y$ and $\bigcup \mathcal{F} \setminus Y \in (Fin \otimes Fin)^{+}$.
- $Y \in Fin \otimes Fin$.
- $\mathcal{F}'$ does not admit a pseudointersection in $(Fin \otimes Fin)^{+}$.
- $\mathfrak{p}(Fin \otimes Fin) = \omega_1$

Theorem(Szymański, Hao Xua)

$\mathfrak{p}(Fin \otimes Fin) = \omega_1$

Corollary

$\mathfrak{p}_{\mathcal{E}_2} = \omega_1$

Since Dobrinen proved that $\mathcal{E}_2$ is a dense subset of $Fin \otimes Fin^{\dotplus}$, this theorem is a consequence of the last theorem.

Theorem (NF)

- For every $n \in \omega$, $\mathfrak{p}_{\mathcal{R}_n} = \mathfrak{p}$.
- For every $n \in \omega$, $\mathfrak{p}_{\mathcal{H}^n} = \mathfrak{p}$.
- If $\mathcal{E}_\infty$ is the Carlson-Simpson space, $\mathfrak{p}_{\mathcal{E}_\infty} \leq \mathfrak{p}$.
- If $\mathcal{R}$ is a topological Ramsey space generate by Fräïssé classes, $\mathfrak{p} \leq \mathfrak{p}_{\mathcal{R}}$.

$*$ For every $k \in \omega$ such that $k > 2$, $\mathfrak{p}_{\mathcal{E}_k} = \omega_1$.

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