Ramsey properties and the Katětov order

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1 Basic Definitions

2 Ramdom graphs ideals

3 Main result

- An ideal on ω is a family $\mathcal{I}$ of subsets of ω closed under finite unions and under subsets. We only consider ideals which contains all finite sets.

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- $\mathcal{I}$ is tall if for every $B \in [\omega]^\omega$ there is $A \in \mathcal{I}$ such that $A \cap B$ is infinite or equivalently for every positive set C the restriction of $\mathcal{I}$ to C is not the *Fin* ideal.

- We will say that $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $f : \omega_{\mathcal{J}} \to \omega_{\mathcal{I}}$ such that $f^{-1}[A] \in \mathcal{I}$ when $A \in \mathcal{J}$.

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- As a last comment, remember that the ramdom graph is the unique (up to isomorfisms) such that for every $a, b \in Fin$ disjoin sets there is $n \in \omega$ related with every $i \in a$ and not related with any $j \in b$.

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- As a last comment, remember that the ramdom graph is the unique (up to isomorfisms) such that for every $a, b \in Fin$ disjoin sets there is $n \in \omega$ related with every $i \in a$ and not related with any $j \in b$. The random graph ideal is the ideal generated by cliques and anticliques and it is denoted by $\mathcal{R}$.

Definition

We say that $\omega \to [\mathcal{I}^+]^2_n$ if for every $c : [\omega]^2 \to n$ there is $A \in \mathcal{I}^+$ such that c is constant in $[A]^2$.

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Observation

The classical Ramsey theorem says that $\omega \to [Fin^+]^2_n$. It's easy to see that $\omega \to [\mathcal{I}^+]^2_2$ if and only if $\mathcal{R} \not\leq_K \mathcal{I}$

Proposition

For every n there is an ideal $\mathcal{R}_n$ such that $\omega \to [\mathcal{I}^+]^2_n$ if and only if $\mathcal{R}_n \not\leq_K \mathcal{I}$

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Proof

- We can construct recursively an universal graph with n colors satisfying a property equivalent with the random graph property. Then we take the ideal generated by monocrhomatic sets and that's all. ■

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- M. Hrušák, D. Meza-Alcántara, E. Thümmel and C. Uzcátegui answered in the negative way giving an example of an ideal $\mathcal{I}$ which satisfies $\omega \to [\mathcal{I}^+]^2_2$ and does not satisfies $\omega \to [\mathcal{I}^+]^2_3$.

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- The main result of this talk is how to improve that result.

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For $s \in \omega^{<\omega}$ we define a family of subsets of ω as follows:

- $A_\emptyset = \omega$
- $\{A_{s^\frown n} : n \in \omega\}$ is a partition of A_s in infinite sets.
- For n, m natural numbers there are $s \neq t \in \omega^{<\omega}$ such that $n \in A_s$ and $m \in A_t$.

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- For $n \geq 1$ define $\widetilde{\mathcal{ED}}^n$ as the ideal generated by A_s such that $|s| = n + 1$ and selectors in every level from 0 to n.

- Note that $\mathcal{R} \leq_K \mathcal{ED}$ or equivalently $\mathcal{ED}$ has no Ramsey properties.

- Also we can define $\widetilde{\mathcal{ED}}^{\omega}$ as the intersection of $\widetilde{\mathcal{ED}}^{n}$ with $n \in \omega$.

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- The result of M. Hrušák, D. Meza-Alcántara, E. Thümmel and C. Uzcátegui can be translated into this langauge as $\omega \rightarrow [\widetilde{\mathcal{ED}}^{1}{}^{+}]^2_2$ but $\widetilde{\mathcal{ED}}^{1} \geq_K R_3$

Main theorem

For every $n \in \omega$ we have that $\omega \rightarrow [\widetilde{\mathcal{ED}}^{n}{}^{+}]^2_{n+1}$ but $\widetilde{\mathcal{ED}}^{n} \geq R_{n+2}$. In other words, we have an example of an ideal such that the Ramsey property happens for some n and fails for $n+1$

Corolary

$\omega \rightarrow [\widetilde{\mathcal{ED}}^{\omega}{}^{+}]^2_{n}$ for every $n \in \omega$.

The idea of the proof

It's easy to see that the Ramsey property with $n+2$ colors is not satisfied by $\widetilde{\mathcal{ED}}^n$ because we can do a coloring by levels. It's a little bit hard to see the Ramsey property with $n+1$ colors, but we have only see that in every node in the first level or we have a positive set for one color or for each color we have infinite monocrhomatic sets.

Question

The main question of this area is if there is a Ramsey Borel ideal. The $\mathcal{ED}$'s ideals seen here was F_σ and the ideal $\widetilde{\mathcal{ED}}^\omega$ is $F_{\sigma\delta}$ but they don't have the strong Ramsey property and in fact every F_σ ideal doesn't have the Ramsey property because they have a restriction bigger (in the Katětov order) than $\mathcal{ED}$. So if it is true then an example should be more complicated.

$\mathcal{THANKS}$

$\mathcal{FOR}$

$\mathcal{YOUR}$

$\mathcal{ATTENTION}$