Ramsey properties and the Katětov order

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1. Basic Definitions

2. Random graphs ideals

3. Main result
An ideal on $\omega$ is a family $\mathcal{I}$ of subsets of $\omega$ closed under finite unions and under subsets. We only consider ideals which contains all finite sets.
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We think an ideal like a family of “small” subsets and it is the dual notion of a filter. If \( A \notin \mathcal{I} \) we say that \( A \) is an \( \mathcal{I} \)-positive set or only a positive set.

\( \mathcal{I} \) is tall if for every \( B \in [\omega]^\omega \) there is \( A \in \mathcal{I} \) such that \( A \cap B \) is infinite or equivalently for every positive set \( C \) the restriction of \( \mathcal{I} \) to \( C \) is not the \( Fin \) ideal.
We will say that $\mathcal{I} \leq_{K} \mathcal{J}$ if there is a function $f : \omega \mathcal{J} \to \omega \mathcal{I}$ such that $f^{-1}[A] \in \mathcal{I}$ when $A \in \mathcal{J}$. 
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As a last comment, remember that the random graph is the unique (up to isomorphisms) such that for every $a, b \in Fin$ disjoin sets there is $n \in \omega$ related with every $i \in a$ and not related with any $j \in b$. 
We will say that $\mathcal{I} \leq_{\mathcal{K}} \mathcal{J}$ if there is a function $f : \omega \mathcal{J} \rightarrow \omega \mathcal{I}$ such that $f^{-1}[A] \in \mathcal{I}$ when $A \in \mathcal{J}$.

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As a last comment, remember that the random graph is the unique (up to isomorfisms) such that for every $a, b \in \text{Fin}$ disjoin sets there is $n \in \omega$ related with every $i \in a$ and not related with any $j \in b$. The random graph ideal is the ideal generated by cliques and anticliques and it is denoted by $\mathcal{R}$. 
Definition

We say that $\omega \rightarrow [\mathcal{I}^+]^2_n$ if for every $c : [\omega]^2 \rightarrow n$ there is $A \in \mathcal{I}^+$ such that $c$ is constant in $[A]^2$. 
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Observation

The classical Ramsey theorem says that $\omega \to [\text{Fin}^+]^2_n$. It’s easy to see that $\omega \to [I^+]^2_2$ if and only if $R \nless_K I$.
Proposition

For every $n$ there is an ideal $\mathcal{R}_n$ such that $\omega \rightarrow [\mathcal{I}^+]^2_n$ if and only if $\mathcal{R}_n \nsubseteq_K \mathcal{I}$
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Proof

- We can construct recursively an universal graph with $n$ colors satisfying a property equivalent with the random graph property. Then we take the ideal generated by monochromatic sets and that’s all. ■
• R. Filipów, N. Mrożek, I. Reclaw and P. Szuca asked if the number of colors matter (in other words if $\omega \rightarrow [\mathcal{I}^+]^2_2$ is equivalent with $\omega \rightarrow [\mathcal{I}^+]^2_n$).
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M. Hrušák, D. Meza-Alcántara, E. Thümmel and C. Uzcátegui answered in the negative way giving an example of an ideal $\mathcal{I}$ which satisfies $\omega \rightarrow [\mathcal{I}^+]^2_2$ and does not satisfies $\omega \rightarrow [\mathcal{I}^+]^2_3$. 
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The main result of this talk is how to improve that result.
Definition

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- $A_\emptyset = \omega$
- $\{A_{s \upharpoonright n} : n \in \omega\}$ is a partition of $A_s$ in infinite sets.
- For $n, m$ natural numbers there are $s \neq t \in \omega^{<\omega}$ such that $n \in A_s$ and $m \in A_t$. 

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February, 2017 8 / 14
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For $n \geq 1$ define $\overline{\mathcal{ED}}^n$ as the ideal generated by $A_s$ such that $|s| = n + 1$ and selectors in every level from 0 to $n$. 
• With the previous definition we can see the $E_D$ ideal as the ideal generated by $A_{(n)}$ and selectors in the first level.

• For $n \geq 1$ define $\widetilde{E_D}^n$ as the ideal generated by $A_s$ such that $|s| = n + 1$ and selectors in every level from 0 to $n$.

• Note that $R \leq_K E_D$ or equivalently $E_D$ has no Ramsey properties.
Also we can define $\overline{\mathcal{ED}}^\omega$ as the intersection of $\overline{\mathcal{ED}}^n$ with $n \in \omega$. 
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The result of M. Hrušák, D. Meza-Alcántara, E. Thümmel and C. Uzcátegui can be translated into this language as $\omega \rightarrow [\overline{ED}^1]^2_2$ but $\overline{ED}^1 \geq_K R_3$. 
Main theorem

For every $n \in \omega$ we have that $\omega \rightarrow [\overline{\mathcal{ED}}^n + ]^2_{n+1}$ but $\overline{\mathcal{ED}}^n \geq R_{n+2}$. In other words, we have an example of an ideal such that the Ramsey property happens for some $n$ and fails for $n + 1$.

Corollary

$\omega \rightarrow [\overline{\mathcal{ED}}^\omega + ]^2_n$ for every $n \in \omega$. 
The idea of the proof

It's easy to see that the Ramsey property with $n + 2$ colors is not satisfied by $\overline{ED}^n$ because we can do a coloring by levels. It's a little bit hard to see the Ramsey property with $n + 1$ colors, but we have only see that in every node in the first level or we have a positive set for one color or for each color we have infinite monochromatic sets.
Question

The main question of this area is if there is a Ramsey Borel ideal. The $\mathcal{ED}$'s ideals seen here was $F_\sigma$ and the ideal $\overline{\mathcal{ED}}^\omega$ is $F_{\sigma\delta}$ but they don’t have the strong Ramsey property and in fact every $F_\sigma$ ideal doesn’t have the Ramsey property because they have a restriction bigger (in the Katětov order) than $\mathcal{ED}$. So if it is true then an example should be more complicated.
THANKS
FOR
YOUR
ATTENTION