Are you self-similar?

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joint work with T. Banakh and F. Strobin
Self-similar sets

$X$ - topological space
$\mathcal{F} = \{ f : X \to X; \text{ continuous, ”contractive” maps} \}$ - finite

\[ \mathcal{F} : \mathcal{P}(X) \to \mathcal{P}(X) \]

$B \subset X \quad \mathcal{F}(B) = \bigcup_{f \in \mathcal{F}} f(B) $
Self-similar sets

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$$\mathcal{F} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$$

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**Definition**

A **self-similar set** for family $\mathcal{F}$ is a nonempty compact set $A \subset X$ which is fixed point of the operator $\mathcal{F}$:

$$A = \mathcal{F}(A) = \bigcup_{f \in \mathcal{F}} f(A).$$
Examples for $X = \mathbb{R}^n$

$F$ : similarities contractive affine transformations

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Examples for $X$-metric space

$F : \begin{align*}
\text{Banach contractions} & \quad \text{weak contractions} \\
\text{Lip}_f < 1 & \quad \forall x \neq y \quad d(f(x), f(y)) < d(x, y)
\end{align*}$

A: IFS-attractor \quad \text{weak IFS-attractor}

Are you self-similar?
Motivation

Which compact space is homeomorphic to
- IFS-attractor in $\mathbb{R}^n$?
- IFS-attractor?
- weak IFS-attractor?

Definition
A compact space $A = F(A)$ for finite $F = \{ f : A \to A, \text{continuous map} \}$ is topological fractal if $A$ is a Hausdorff space and $F$ is topologically contracting; for every open cover $U$ of $A$ there is $n \in \mathbb{N}$ such that for any maps $f_1, \ldots, f_n \in F$ the set $f_1 \circ \cdots \circ f_n(A) \subset U \in U$. 

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Motivation

Which compact space is homeomorphic to

- IFS-attractor in $\mathbb{R}^n$? $\rightarrow$ Euclidean fractal
- IFS-attractor?
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Motivation

Which compact space is homeomorphic to
- IFS-attractor in $\mathbb{R}^n$? → **Euclidean fractal**
- IFS-attractor? → **Banach fractal**
- weak IFS-attractor? → **topological fractal**
Motivation

Which compact space is homeomorphic to

- IFS-attractor in \( \mathbb{R}^n \)? \(\rightarrow\) **Euclidean fractal**
- IFS-attractor? \(\rightarrow\) **Banach fractal**
- weak IFS-attractor? \(\rightarrow\) **topological fractal**

**Euclidean fractal \(\Rightarrow\) Banach fractal \(\Rightarrow\) topological fractal**
Motivation

Which compact space is homeomorphic to

- IFS-attractor in \( \mathbb{R}^n \)? \(\rightarrow\) **Euclidean fractal**
- IFS-attractor? \(\rightarrow\) **Banach fractal**
- weak IFS-attractor? \(\rightarrow\) **topological fractal**

Euclidean fractal \(\Rightarrow\) Banach fractal \(\Rightarrow\) topological fractal

**Definition**

A compact space \( A = \mathcal{F}(A) \) for finite \( \mathcal{F} = \{ f : A \to A, \text{continuous map} \} \) is **topological fractal** if \( A \) is a Hausdorff space and \( \mathcal{F} \) is **topologically contracting**; for every open cover \( \mathcal{U} \) of \( A \) there is \( n \in \mathbb{N} \) such that for any maps \( f_1, \ldots, f_n \in \mathcal{F} \) the set \( f_1 \circ \cdots \circ f_n(A) \subset U \in \mathcal{U} \).
Peano continua

Peano continuum = continuous image of $[0, 1]$
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**Corollary from (Hata, 1985)**

For compact and connected set $A$

$A$ is a Banach fractal $\Rightarrow$ $A$ is a Peano continuum
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Corollary from (Hata, 1985)

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Is every Peano continuum a topological fractal?

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Peano continuum = continuous image of $[0, 1]$

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Is every Peano continuum a topological fractal?
Theorem

Each Peano continuum $P$ with open subset $A$ homeomorphic to $(0, 1)$ is a topological fractal.

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Theorem

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1. $\overline{A}$ is the IFS-attractor of $\mathcal{F}$
Peano continua

Theorem

Each Peano continuum $P$ with open subset $A$ homeomorphic to $(0,1)$ is a topological fractal.

1. $\overline{A}$ is the IFS-attractor of $\mathcal{F}$
2. For every $f \in \mathcal{F}$
   \[ \tilde{f}|_A = f \text{ and } \tilde{f}|_{P \setminus A} = \text{const}_{x_f} \]
   \[ \tilde{f} : P \to \overline{A} \text{ for } f \in \mathcal{F} \text{ and } \bigcup_{f \in \mathcal{F}} \tilde{f}(P) = \overline{A} \]
Theorem

Each Peano continuum $P$ with open subset $A$ homeomorphic to $(0,1)$ is a topological fractal.

1. $\overline{A}$ is the IFS-attractor of $F$
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   \[ \tilde{f}|_A = f \quad \text{and} \quad \tilde{f}|_{P \setminus A} = \text{const}_{x_f} \]
   \[ \tilde{f} : P \rightarrow \overline{A} \quad \text{for} \quad f \in F \quad \text{and} \quad \bigcup_{f \in F} \tilde{f}(P) = \overline{A} \]
3. $g(\overline{A}) = P \setminus A$ and $g|_{P \setminus A} = \text{const}_{x_g}$
   \[ g : P \rightarrow P \setminus A \quad \text{and} \quad g(P) = P \setminus A. \]
Theorem

Each Peano continuum $P$ with open subset $A$ homeomorphic to $(0,1)$ is a topological fractal.

1. $\bar{A}$ is the IFS-attractor of $\mathcal{F}$

2. For every $f \in \mathcal{F}$
   \[
   \tilde{f}_{|\bar{A}} = f \quad \text{and} \quad \tilde{f}_{|P \setminus A} = \text{const}_{x_f}
   \]
   \[
   \tilde{f} : P \rightarrow \bar{A} \quad \text{for} \quad f \in \mathcal{F} \quad \text{and} \quad \bigcup_{f \in \mathcal{F}} \tilde{f}(P) = \bar{A}
   \]

3. $g(\bar{A}) = P \setminus A$ and $g_{|P \setminus A} = \text{const}_{x_g}$
   \[
   g : P \rightarrow P \setminus A \quad \text{and} \quad g(P) = P \setminus A.
   \]

4. $g$ is uniformly continuous so $\hat{\mathcal{F}} \cup \{g\}$ is topologically contracting
Zero-dimensional spaces

**Theorem (Banakh, N., Strobin, 2015)**

For zero-dimensional space $X$

- $X$ countable and $ht(X) = \alpha + 1 \Rightarrow X$ is an Euclidean fractal
- $X$ countable and $ht(X)$ - limit ordinal $\Rightarrow X$ is not a topological fractal
- $X$ uncountable $\Rightarrow X$ is an Euclidean fractal
Theorem (Banakh, N 2016)

Let $X$ be compact finite-dimensional space and $Z$ be its uncountable, zero-dimensional, subset open in $X$. Then $X$ is topological fractal.

Theorem (Banakh, N, Strobin)

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In progress...
Compact spaces

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Let $X$ be compact finite-dimensional space and $Z$ be its uncountable, zero-dimensional, subset open in $X$. Then $X$ is Euclidean fractal.

In progress...
Golden Bee (Robert Ammann)

\[ (x, y) = (0, a - a_0)(x, y) + (0, a) \]
\[ (x, y) = (-a_0^2, 0)(x, y) + (1, 0) \]

\[ a = \sqrt{r} \] where \( r = \frac{\sqrt{5} - 1}{2} \) - golden ratio conjugate

Golden Bee (Robert Ammann)

\[ f_1(x, y) = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ a \end{pmatrix} \]

\[ f_2(x, y) = \begin{pmatrix} -a^2 & 0 \\ 0 & a^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ a = \sqrt{r} \text{ where } r = \frac{\sqrt{5} - 1}{2} \text{ - golden ratio conjugate} \]

Golden Bee tilings

$$f_1(G)$$

$$f_2(G)$$

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Golden Bee tilings

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Are you self-similar?
**Golden Bee - amount of pieces**

<table>
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<tr>
<th></th>
<th>Large: 1</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5 ...</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Small: 0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3 ...</td>
</tr>
</tbody>
</table>

The Fibonacci sequence is defined as:

\[ F_n = F_{n-1} + F_{n-2} \]

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Golden Bee - amount of pieces

Large: 1 1 2 3 5 ...
Small: 0 1 1 2 3 ...

\[ F_n = F_{n-1} + F_{n-2} \]

Fibonacci sequence
Let’s play!

Take all of the bee-shaped tiles and fit them together to make a large bee.
Self-similar polygonal tilings

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THANK YOU