Are you self-similar?

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joint work with T. Banakh and F. Strobin

Self-similar sets

X - topological space
$\mathcal{F} = \{f : X \to X;$ continuous, "contractive" maps$\}$ - finite

$$\mathcal{F} : \mathcal{P}(X) \to \mathcal{P}(X)$$

$$B \subset X \qquad \mathcal{F}(B) = \bigcup_{f \in \mathcal{F}} f(B)$$

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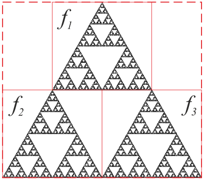
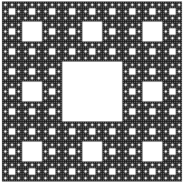
Definition

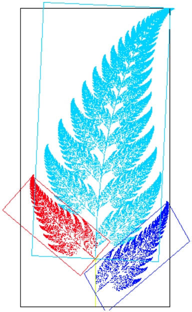
A **self-similar set** for family $\mathcal{F}$ is a nonempty compact set $A \subset X$ which is fixed point of the operator $\mathcal{F}$:

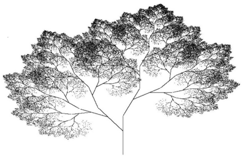
$$A = \mathcal{F}(A) = \bigcup_{f \in \mathcal{F}} f(A).$$

$\mathcal{F}$: similarities contractive affine transformations



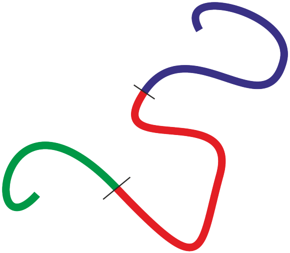


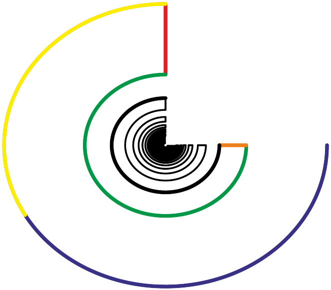




Examples for X- metric space

$\mathcal{F}$: Banach contractions $\text{Lip} f < 1$ | weak contractions $\forall x \neq y \; d(f(x), f(y)) < d(x, y)$





A: IFS-attractor | weak IFS-attractor

Motivation

Which compact space is homeomorphic to

- IFS-attractor in $\mathbb{R}^n$?
- IFS-attractor?
- weak IFS-attractor?

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- weak IFS-attractor? $\rightarrow$ **topological fractal**

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Euclidean fractal $\Rightarrow$ Banach fractal $\Rightarrow$ topological fractal

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Euclidean fractal $\Rightarrow$ Banach fractal $\Rightarrow$ topological fractal

Definition

A compact space $A = \mathcal{F}(A)$ for finite $\mathcal{F} = \{f\colon A \to A$, continuous map$\}$ is **topological fractal** if A is a Hausdorff space and $\mathcal{F}$ is *topologically contracting*; for every open cover $\mathcal{U}$ of A there is $n \in \mathbb{N}$ such that for any maps $f_1, \ldots, f_n \in \mathcal{F}$ the set $f_1 \circ \cdots \circ f_n(A) \subset U \in \mathcal{U}$.

Peano continua

Peano continuum $=$ continuous image of $[0,1]$

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Corollary from (Hata, 1985)

For compact and connected set A

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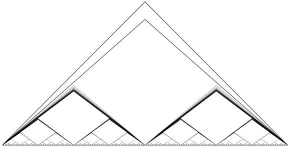

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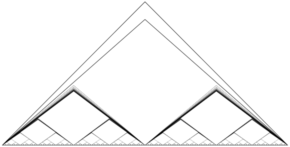
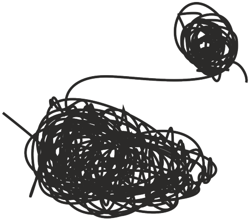


Is every Peano continuum a topological fractal?

Theorem

Each Peano continuum P with open subset A homeomorphic to $(0,1)$ is a topological fractal.



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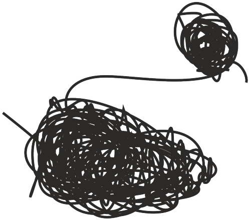

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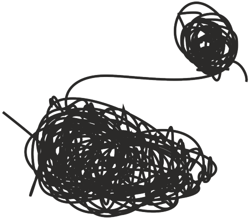

1. $\overline{A}$ is the IFS-attractor of $\mathcal{F}$
2. For every $f \in \mathcal{F}$
   $\tilde{f}_{|\overline{A}} = f$ and $\tilde{f}_{|P\setminus A} = const_{x_f}$
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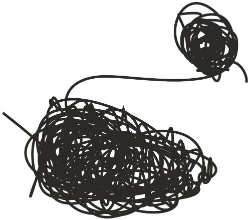

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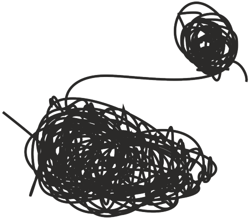

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   $g: P \to P \setminus A$ and $g(P) = P \setminus A$.
4. g is uniformly continuous so $\tilde{\mathcal{F}} \cup \{g\}$ is topologically contracting

Zero-dimensional spaces

Theorem (Banakh, N. , Strobin, 2015)

For zero-dimensional space X

- X countable and $ht(X) = \alpha + 1 \Rightarrow X$ is an Euclidean fractal
- X countable and $ht(X)$ - limit ordinal $\Rightarrow X$ is not a topological fractal
- X uncountable $\Rightarrow X$ is an Euclidean fractal

Compact spaces

Theorem (Banakh, N 2016)

Let X be compact finite-dimensional space and Z be its uncountable, zero-dimensional, subset open in X. Then X is topological fractal.

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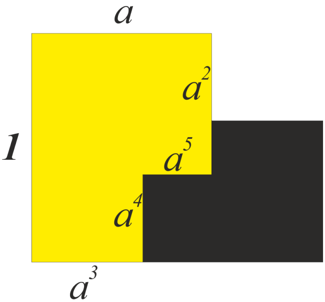
In progress...

Golden Bee (Robert Ammann)



$a = \sqrt{r}$ where $r = \frac{\sqrt{5}-1}{2}$ - golden ratio conjugate

B. Grunbaum and G. S. Shephard, *Tilings and Patterns*, Freeman, New York, NY, 1987.

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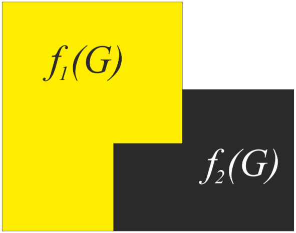

$$f_1(x,y) = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ a \end{pmatrix}$$

$$f_2(x,y) = \begin{pmatrix} -a^2 & 0 \\ 0 & a^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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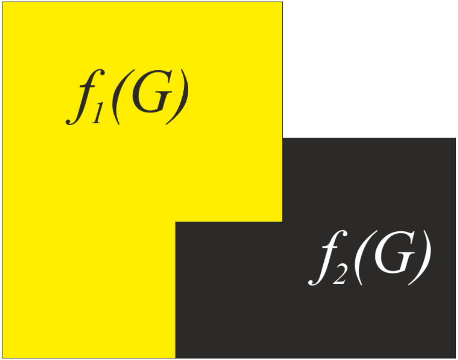
$f_1(G)$
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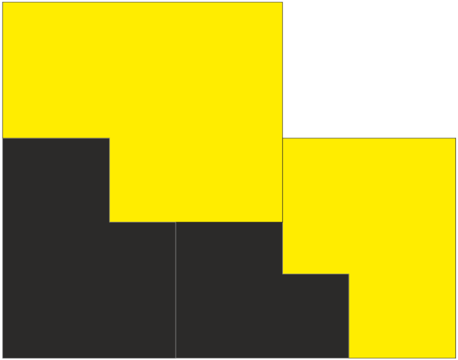
Golden Bee tilings



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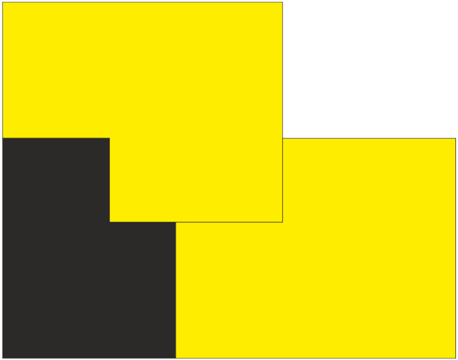

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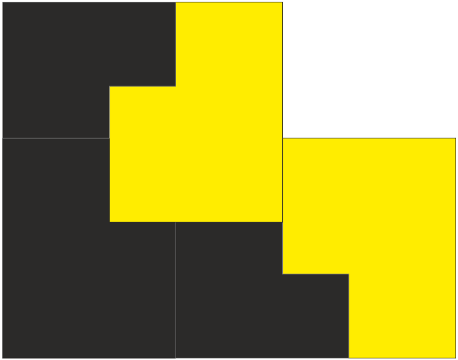

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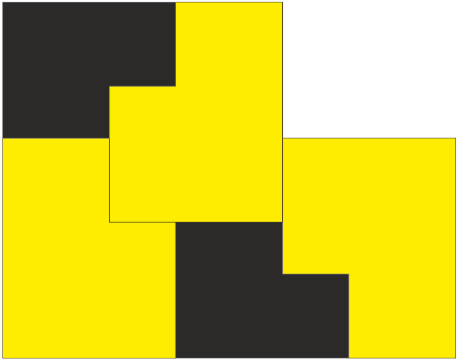

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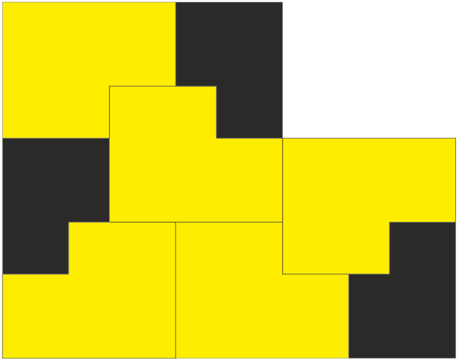
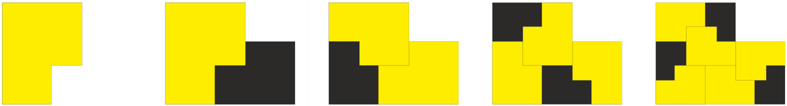


Golden Bee - amount of pieces



Large: 1	1	2	3	5 ...
Small: 0	1	1	2	3 ...

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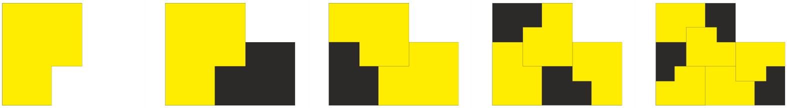

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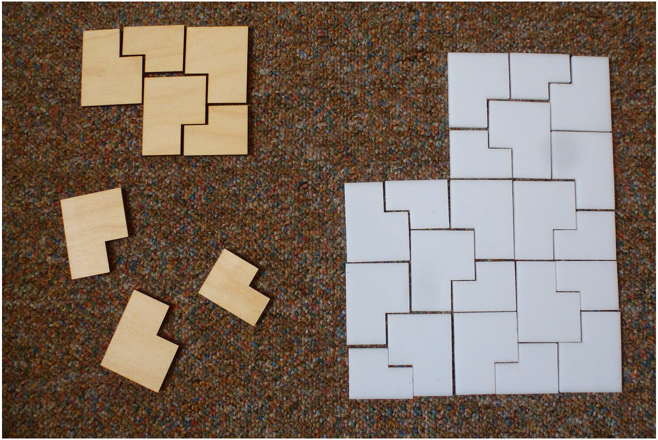
$$F_n = F_{n-1} + F_{n-2}$$

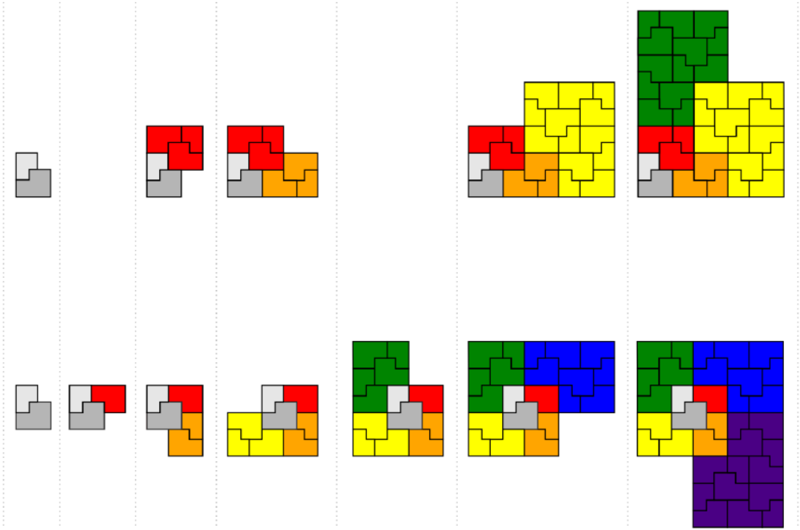
Fibonacci sequence

Let's play!

Take all of the bee-shaped tiles ant fit them together to make a large bee.



Self-similar polygonal tilings



THANK YOU