Characterizing chainable and tree-like continua

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Chainable continua

Continuum = compact connected Hausdorff space

Definition 1
An open cover $\mathcal{U}$ of $X$ is called chain-open cover of $X$ if for $\mathcal{U}$ there is an enumeration $\mathcal{U} = \{U_1, \ldots, U_m\}$ such that

$$U_i \cap U_j \neq \emptyset \iff |i - j| \leq 1 \text{ for all } 1 \leq i, j \leq n.$$ 

Definition 2
A continuum $X$ is called chainable if each open cover of $X$ there is a chain-open cover refinement.

Remark
In metric case a continuum $X$ is chainable iff for each $\varepsilon > 0$ there is a chain-open cover $\mathcal{U}$ of $X$ with mesh($\mathcal{U}$) $< \varepsilon$. 
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A continuum $X$ is called $n$-chainable, where $n \geq 3$, if for any open cover $\mathcal{U}$ with $|\mathcal{U}| \leq n$ there is an open-chain refinement.

A continuum $X$ is a chainable iff $X$ is $n$-chainable for each $n$.

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If $X$ is $n$-chainable, then $X$ is $m$-chainable for each $m \leq n$.

In the paper T. Banakh, P. Bankston, B. Raines, W. Ruitenburg, *Chainability and Hemmingsen’s theorem*, Topology Appl. 153 (2006) 2462–2468, it was announced (without proof) that 4-chainable implies $n$-chainability for each $n$. In other words 4-chainability implies chainability.

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Let \( \mathcal{U} = \{ U_1, \ldots, U_n \} \) be a finite family of subsets of an arbitrary space \( X \) and let \( p_1, \ldots, p_n \) be a system of points of an euclidean space \( \mathbb{R}^m \). A nerve of the family \( \mathcal{U} \) is the simplicial complex \( \mathcal{N}(\mathcal{U}) \) formed by simplexes \( \langle p_{i_0}, \ldots, p_{i_l} \rangle \) such that \( U_{i_0} \cap \cdots \cap U_{i_l} \neq \emptyset \).

**Theorem 1**

If \( X \) is a normal space and \( \mathcal{U} = \{ U_1, \ldots, U_n \} \) is a finite open cover of \( X \) then there is a map from \( X \) into the nerve \( \mathcal{N}(\mathcal{U}) \) i.e. there is a continuous map \( \kappa: X \to |\mathcal{N}(\mathcal{U})| \) such that

\[
\kappa^{-1}(\text{st}_{|\mathcal{N}(\mathcal{U})|} p_i) \subseteq U_i
\]

for every \( i = 1, \ldots, n \).

\( |\mathcal{N}(\mathcal{U})| = \bigcup \mathcal{N}(\mathcal{U}) \) – a carrier of the nerve \( \mathcal{N}(\mathcal{U}) \);

\( \text{st}_{|\mathcal{N}(\mathcal{U})|} p_i = |\mathcal{N}(\mathcal{U})| \setminus \bigcup \{ S \in \mathcal{N}(\mathcal{U}) : p_i \notin S \} \) – a star of vertex \( p_i \) in \( |\mathcal{N}(\mathcal{U})| \).
\textbf{Definition 4}

Let $\mathcal{U}$ be an open cover of a topological space $X$. A map $f : X \to Y$ into a space $Y$ is called a \textit{$\mathcal{U}$-map} if there is an open cover $\mathcal{V}$ of $Y$ whose preimage $f^{-1}(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\}$ refines $\mathcal{U}$.

\textbf{Remark}

Let $X$ and $Y$ be a compact Hausdorff spaces and let $\mathcal{U}$ be an open cover of $X$. A map $f : X \to Y$ is $\mathcal{U}$-map iff for each $y \in Y$ there is $U \in \mathcal{U}$ such that $f^{-1}(y) \subseteq U$.

\textbf{Theorem 2}

\textit{If $X$ is a normal space then $\dim X = 1$ iff for any open cover $\mathcal{U}$ of $X$ there is a $\mathcal{U}$-map $f : X \to \Gamma$ onto a graph $\Gamma$.}

Graph = carrier of 1-dimensional simplicial complex.
If $\mathcal{U}$ is a chain then the carrier $|\mathcal{N}(\mathcal{U})|$ of the nerve $\mathcal{N}(\mathcal{U})$ is an arc, so:

**Theorem 3**

1. A continuum $X$ is chainable if and only if for any (finite) open cover $\mathcal{U}$ of $X$ there is a $\mathcal{U}$-map from $X$ onto an arc.

2. A continuum $X$ is $n$-chainable iff for each open cover $\mathcal{U}$ of $X$ such that $|\mathcal{U}| \leq n$ there is a $\mathcal{U}$-map $f$ from $X$ onto an arc.
Tree-like continua

A continuum $X$ is said to be tree-like provided that every open cover of $X$ can be refined by a finite open cover having nerve a tree, that is, having nerve a connected acyclic graph. Similarly, to the $n$-chainability we define the notion of a $n$-tree-likeness.

A counterpart of the Theorem 3 for the class of tree-like continua is the following:

Theorem 4

1. A continuum $X$ is a tree-like if and only if for each open cover $\mathcal{U}$ of $X$ there is a tree $T$ and $\mathcal{U}$-map $f : X \to T$.

2. A continuum $X$ is a $n$-tree-like if and only if for each open cover $\mathcal{U}$ of $X$ such that $|\mathcal{U}| \leq n$ there is a tree $T$ and $\mathcal{U}$-map $f : X \to T$. 
A continuum $X$ is said to be *tree-like* provided that every open cover of $X$ can be refined by a finite open cover having nerve a tree, that is, having nerve a connected acyclic graph. Similarly, to the $n$-chainability we define the notion of a $n$-tree-likeness.

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Chainable and tree-like continua are particular cases of the concept of a $\mathcal{X}$-like continuum. We shall say that a continuum $X$ is $\mathcal{X}$-like, where $\mathcal{X}$ is a class of continua, if for any open cover $U$ of $X$ there is a $U$-map $f : X \to T$ onto some space $T \in \mathcal{X}$.

A continuum $X$ is called $n$-$\mathcal{X}$-like is for any open cover $U$ with $|U| \leq n$ there is an $U$-map $f : X \to T \in \mathcal{X}$. 
Chainable and tree-like continua are particular cases of the concept of a $\mathcal{T}$-like continuum. We shall say that a continuum $X$ is $\mathcal{T}$-like, where $\mathcal{T}$ is a class of continua, if for any open cover $\mathcal{U}$ of $X$ there is a $\mathcal{U}$-map $f : X \to T$ onto some space $T \in \mathcal{T}$.

A continuum $X$ is called $n$-$\mathcal{T}$-like is for any open cover $\mathcal{U}$ with $|\mathcal{U}| \leq n$ there is an $\mathcal{U}$-map $f : X \to T \in \mathcal{T}$. 
Main Theorem

Theorem 5

For a subclass $\mathcal{C}$ of the class of tree-like continua and a continuum $X$ the following conditions are equivalent:

1. $X$ is $\mathcal{C}$-like (i.e. for each open cover $\mathcal{U}$ of $X$ there is a $\mathcal{U}$-map $f: X \to T$ of $X$ onto a space $T \in \mathcal{C}$);

2. $X$ is 4-$\mathcal{C}$-like (i.e for each 4-set open cover $\mathcal{U} = \{U_1, U_2, U_3, U_4\}$ of $X$ there is a $\mathcal{U}$-map $f: X \to T$ of $X$ onto a space $T \in \mathcal{C}$).

Corollary 1

A continuum $X$ is chainable (resp. tree-like) if and only if each 4-set open cover $\mathcal{U} = \{U_1, U_2, U_3, U_4\}$ of $X$ has a chain-open (resp. tree-open) refinement.
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Theorem 5

For a subclass \( \mathcal{T} \) of the class of tree-like continua and a continuum \( X \) the following conditions are equivalent:

1. \( X \) is \( \mathcal{T} \)-like (i.e. for each open cover \( \mathcal{U} \) of \( X \) there is a \( \mathcal{U} \)-map \( f : X \to T \) of \( X \) onto a space \( T \in \mathcal{T} \));

2. \( X \) is 4-\( \mathcal{T} \)-like (i.e. for each 4-set open cover \( \mathcal{U} = \{U_1, U_2, U_3, U_4\} \) of \( X \) there is a \( \mathcal{U} \)-map \( f : X \to T \) of \( X \) onto a space \( T \in \mathcal{T} \)).

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A continuum \( X \) is chainable (resp. tree-like) if and only if each 4-set open cover \( \mathcal{U} = \{U_1, U_2, U_3, U_4\} \) of \( X \) has a chain-open (resp. tree-open) refinement.
Theorem 6 (Hemmingsen)

For a compact Hausdorff space $X$ the following conditions are equivalent:

1. $\dim X \leq 1$, which means that any open cover $\mathcal{U}$ of $X$ has an open refinement $\mathcal{V}$ of order $\leq 2$;

2. each 3-set open cover $\mathcal{U}$ of $X$ has an open refinement $\mathcal{V}$ of order $\leq 2$;

3. each 3-set open cover $\mathcal{U} = \{U_1, U_2, U_3\}$ of $X$ has an open 3-set refinement $\mathcal{V} = \{V_1, V_2, V_3\}$ with $V_1 \cap V_2 \cap V_3 = \emptyset$;

Corollary 2

Let $\mathcal{S}$ be a some class of tree-like continua. If $X$ is a $n\mathcal{S}$-like continuum, $n \geq 3$, then $\dim X = 1$. 
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Corollary 2

Let $\mathcal{C}$ be a some class of tree-like continua. If $X$ is a $n$-$\mathcal{C}$-like continuum, $n \geq 3$, then $\dim X = 1$. 
2. Auxiliary facts

Lemma 1

For any open cover $\mathcal{U}$ of a topological graph $\Gamma$ there is a $\mathcal{U}$-map $f : \Gamma \to G$ onto a topological graph of degree $\leq 3$.

This lemma can be easily proved by induction with respect to number of branching point of $\Gamma$. The following drawing illustrates how to decrease a degree of a selected vertex of a graph.

\begin{center}
\includegraphics[width=\textwidth]{lemma_diagram.png}
\end{center}
Lemma 1

For any open cover \( \mathcal{U} \) of a topological graph \( \Gamma \) there is a \( \mathcal{U} \)-map \( f : \Gamma \to G \) onto a topological graph of degree \( \leq 3 \).

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In the next lemma graph \( G \) is consider as a combinatorial object.

**Lemma 2**

*Let \( G = (V, E) \) be a connected graph with \( \deg(G) \leq 3 \) such that \( d(u, v) \geq 6 \) for any two vertices \( u, v \in V \) of order 3. Then there is a 4-coloring \( \chi: V \to \{1, 2, 3, 4\} \) such that no distinct vertices \( u, v \in V \) with \( d(u, v) \leq 2 \) have the same color.*

**Proof.**

Let \( V_3 = \{v \in V : \deg(v) = 3\} \) and let \( B(v) = \{u \in V : \{u, v\} \in E\} \) for each \( v \in V \). Since \( \deg(G) \leq 3 \) then \( |B(v)| \leq 4 \) for each \( v \in V \). Moreover:

\[
v, w \in V_3, v \neq w \Rightarrow B(v) \cap B(w) = \emptyset.
\]

So we can define a 4-coloring \( \chi \) on the union \( \bigcup_{v \in V_3} B(v) \) so that \( \chi \) is injective on each \( B(v) \) and \( \chi(v) = \chi(w) \) for each \( v, w \in V_3 \). Next, it remains to color the remaining vertices all of order \( \leq 2 \) by four colors \( \chi(x) \neq \chi(y) \) if \( d(x, y) \leq 2 \). It is easy to check that this always can be done. \( \square \)
Take a class $\mathcal{T}$ of tree-like continua and assume that $X$ is a continuum such that for any 4-set open cover $\mathcal{U}_4$ of $X$ there is a $\mathcal{U}_4$-map $f : X \to T$ onto a space $T \in \mathcal{T}$. We should prove that such a map exists for any (finite) open cover $\mathcal{U}$ of $X$.

By the corollary 2, there is a $\mathcal{U}$-map $f : X \to \Gamma$ onto a topological graph $\Gamma$. Because of Lemma 1, we can assume that $\text{deg}(\Gamma) \leq 3$. Selecting vertices on edges of $\Gamma$, we find so fine triangulation $G = (V, E)$ of the topological graph $\Gamma$ that

- the distance between any vertices of order 3 in the path metric of $G$ is $\geq 6$;
- the cover $\{f^{-1}(\text{st}_\Gamma(v)) : v \in V\}$ of $X$ is inscribed into $\mathcal{U}$.

Lemma 2 implies that there is a 4-coloring $\chi : V \to \{1, 2, 3, 4\}$ of $V$ such that no vertices $u, v \in V$ with $0 < d(u, v) \leq 2$ are monochromatic.
For $i \in \{1, 2, 3, 4\}$ consider the open set

$$V_i = \bigcup \{\text{st}_\Gamma v : \chi(v) = i\}. $$

The sets $V_1, \ldots, V_4$ cover $\Gamma$. Then for the 4-set cover $\mathcal{V} = \{f^{-1}(V_i) : i \leq 4\}$ of $X$ we can find a $\mathcal{V}$-map $g : X \to Y$ to a continuum $Y \in \mathcal{F}$.

Since $Y$ is tree-like, there is a map $\pi : Y \to T$ onto a topological tree such that the composition $h = \pi \circ g : X \to T$ still is a $\mathcal{V}$-map. Take a triangulation $H = (V_T, E_T)$ of $T$ so fine that the cover $\{h^{-1}(\text{st}_T t) : t \in V_T\}$ is inscribed into the cover $\mathcal{V}$. Consequently, for each vertex $t \in V_T$ we can find a number $\xi(t) \in \{1, 2, 3, 4\}$ such that

$$h^{-1}(\text{st}_T t) \subseteq f^{-1}(V_{\xi(t)}).$$

Using the property of the coloring $\chi$ we can prove that

$$(\forall t \in V_T)(\exists! v_t \in V)(\xi(t) = \chi(v_t)).$$

So,

$$g^{-1}(\pi^{-1}(\text{st}_T t)) = h^{-1}(\text{st}_T t) \subseteq f^{-1}(\text{st}_\Gamma v_t) \subseteq U$$

for some $U \in \mathcal{U}$. It means that $g$ is $\mathcal{U}$-map, so this finishes the proof.
Theorem 7

For a 1-dimensional continuum $X$ the following conditions are equivalent:

1. each map from $X$ into the circle is homotopic to a constant map;
2. $X$ is 3-chainable;
3. $X$ is 3-tree-like.

chainable continua $⊊$ tree-like continua $⊊$ 3-chainable continua $= 3$-tree-like continua

There is a 1-dimensional continuum for which each map from $X$ into the circle is homotopic to a constant map but it is not tree-like – J. H. Case, R. E. Chamberlin Characterizations of tree-like continua, Pacific J. Math. 10 (1960) 73–84.