Linearly ordered Splitting families

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Introduction

Let S and X be infinite subsets of ω. We say that S *splits* X if $S \cap X$ and $X \setminus S$ are both infinite. A family $\mathcal{S} \subseteq [\omega]^{\omega}$ is called an *splitting family* if for every $X \in [\omega]^{\omega}$ there is $S \in \mathcal{S}$ such that S splits X. We will say a family is *linearly ordered* if it is linearly ordered under the almost inclusion (recall that A is an *almost subset of* B (denoted by $A \subseteq^* B$) if $A \setminus B$ is finite).

Problem

Are there linearly ordered splitting families?

Note that if $\mathcal{S}$ is linearly ordered splitting family then:

1. $\mathcal{S}$ does not have a smallest or largest element.
2. There are no inmmediate succesors, in particular it can not be a well order.

We can construct such families assuming the Continuum Hypothesis.

Definition

Let κ and λ be two regular cardinal numbers. We say $\mathcal{G} = (\mathcal{A}, \mathcal{B})$ is a (κ, λ)-*pregap* if the following holds:

1. $\mathcal{A}$ is an increasing family (under the almost inclusion) of size κ.
2. $\mathcal{B}$ is a decreasing family (under the almost inclusion) of size λ.
3. If $A \in \mathcal{A}$ and $B \in \mathcal{B}$ then $A \subseteq^* B$.

A pregap $\mathcal{G} = (\mathcal{A}, \mathcal{B})$ is a *gap* if it *can not be filled* (i.e. there is no $X \in [\omega]^\omega$ such that $A \subseteq^* X \subseteq^* B$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$). The construction (under CH) of a linearly ordered splitting family can be easily done with the following result:

Lemma (Rothberger, Hausdorff)

There are no (κ, λ)*-gaps where* $\kappa, \lambda \in \{0, 1, \omega\}$.

Antonio Aviles and Felix Cabello constructed interesting Banach spaces assuming the existence of a linearly ordered splitting family. This lead them to ask the following:

Problem (Aviles, Cabello)

Is the existence of a linearly ordered splitting family consistent with the failure of CH*?*

Frequently, a "CH construction" can be realized assuming that certain cardinal invariant is equal to $\mathfrak{c}$ (the cardinality of 2^{ω}). In this case the natural cardinal invariant would be the following:

Definition

Let $\mathfrak{j}$ be the least κ for which there is a (κ, κ)-gap.

Obviously, a linearly ordered splitting family can be constructed assuming $\mathfrak{j} = \mathfrak{c}$. However we did not get anything new:

Theorem (Hausdorff)

There is a (ω_1, ω_1)-gap (i.e. $\mathfrak{j} = \omega_1$)

Hence, a straightforward generalization of the previous argument can not be done if $\mathfrak{c} > \omega_1$.

A natural attempt to solve the problem, would be to construct a Sacks indestructible linearly ordered splitting family. However, this idea is also doom to fail because of the following result:

Theorem

Every linearly ordered splitting family has size continuum.

Definitions

Let $\mathcal{S}$ be a linearly ordered family and $\mathcal{G} = (\mathcal{A}, \mathcal{B})$ a pregap.

1. We say $\mathcal{G}$ is a *tight pregap* if there is no $X \in [\omega]^{\omega}$ such that the following holds:
    1. $X \subseteq^* B$ for every $B \in \mathcal{B}$.
    2. $X \cap A$ is finite for every $A \in \mathcal{A}$.
2. We say $\mathcal{G}$ is a *cut of* $\mathcal{S}$ if ($\mathcal{G}$ is a pregap) and $\mathcal{S} = \mathcal{A} \cup \mathcal{B}$.

We can then prove the following:

Lemma

Let $\mathcal{S}$ be a linearly ordered family. The following are equivalent:

1. *$\mathcal{S}$ is splitting.*
2. *Every cut of $\mathcal{S}$ is a non tight pregap.*

Theorem

There is a linearly ordered splitting family in the Cohen model.

To prove the previous result we need the following definition:

Lemma

Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{B} \rangle$ *be a pregap. We define the forcing* $\mathbb{P}(\mathcal{G})$ *as the set of all* $p = (s_p, L_p, R_p)$ *where* $s_p \in [\omega]^{<\omega}$, $L_p \in [\mathcal{A}]^{<\omega}$, $R_p \in [\mathcal{B}]^{<\omega}$ *and* $\Delta(L_p, R_p) = \{\Delta(A, B) \mid A \in L_p \wedge B \in R_p\} \subseteq max(s_p)$. *If* $p, q \in \mathbb{P}(\mathcal{A}, \mathcal{B})$ *then* $p \leq q$ *if the following holds:*

1. $s_q \sqsubseteq s_p$, $L_q \subseteq L_p$, $R_q \subseteq R_p$.
2. *If* $max(s_q) < i \leq max(s_p)$ *then:*
    1. *If* $i \in \bigcup L_q$ *then* $i \in s_p$.
    2. *If* $i \notin \bigcap R_q$ *then* $i \notin s_p$.

Regarding the non existence we have the following:

Theorem

OCA + $\mathfrak{p} > \omega_1$ *implies that there are no linearly ordered splitting families.*

We know that $\mathfrak{p} > \omega_1$ is not enough for the following result, but we do not know if OCA suffies to destroy such families.

Problem

Does OCA *implies that there are no linearly ordered splitting families?*