Interplay between two generalizations of the first countability

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(Lviv and Kielce)

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First-countable spaces

Definition

A topological space X is *first-countable* if at each point x the space X has a countable neighborhood base, i.e., a countable family $\mathcal{B}_x$ of open sets such that for any neighborhood $O_x \subset X$ of x there is a set $B \in \mathcal{B}_x$ such that $x \in B \subset O_x$.

We shall discuss the interplay between two generalizations of the first-countability.

The first generalization replaces countable neighborhood bases $\mathcal{B}_x$ by neighborhood bases $\{B_p\}_{p \in P}$ indexed by some partially ordered sets P, more complicated than ω.

The second generalization replaces countable neighborhood bases by countable networks with some additional properties.

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Neighborhood P-bases

Definition

Let $(P, \leq)$ be a partially ordered set. A neighborhood base $\mathcal{B}_x$ at a point x of a topological space X is called a *neighborhood P-base* if $\mathcal{B}_x$ admits a monotone enumeration $\mathcal{B}_x = \{B_p\}_{p \in P}$, which means that $B_q \subset B_p$ for any $p \leq q$ in P.

A topological space X is first-countable if and only if at each point $x \in X$ the space X has a neighborhood ω-base $\mathcal{B}_x$.

Problem

What can be said about spaces possessing a neighborhood ω^ω-base at each point?
Here ω^ω is the set of all functions from ω to ω, endowed with the coordinatewise partial order.

In literature ω^ω-bases are called $\mathfrak{G}$-bases.

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Spaces with a neighborhood ω^ω-base need not be first-countable.

Example

The countable box-product $\square_{n\in\omega} X_n$ of first-countable spaces has a neighborhood ω^ω-base at each point.

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cs^* and s^*-networks

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A family $\mathcal{N}$ of subsets of a topological space X is called

- a cs^*-*network* at a point $x \in X$ if for any neighborhood $O_x \subset X$ of x and any sequence $\{x_n\}_{n\in\omega} \subset X$ convergent to x there exists a set $N \in \mathcal{N}$ such that $x \in N \subset O_x$ and N contains infinitely many points x_n, $n \in \omega$;
- an s^*-*network* at a point $x \in X$ if for any neighborhood $O_x \subset X$ of x and any sequence $\{x_n\}_{n\in\omega} \subset X$ accumulating at x there exists a set $N \in \mathcal{N}$ such that $x \in N \subset O_x$ and N contains infinitely many points x_n, $n \in \omega$.

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A topological space X has

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first-countable $\Leftrightarrow$ countable character $\Rightarrow$ countable s*-character $\Rightarrow$ countable cs*-character

What is the relation of these properties to ω^ω-bases?

Network generalizations of first-countability

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Main Theorem

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If a topological space X has a neighborhood ω^ω-base at a point $x \in X$, then X has a countable s*-network at x.

Proof.

Let $\{U_\alpha\}_{\alpha\in\omega^\omega}$ be a neighborhood ω^ω-base
(so $U_\beta \subset U_\alpha$ for all $\alpha \le \beta$ in ω^ω).
For a subset $A \subset \omega^\omega$ put $U_A := \bigcap_{\alpha\in A} U_\alpha$.
Observe that ω^ω carries a natural Polish topology generated by the countable base $\{{\uparrow}\alpha\}_{\alpha\in\omega^{<\omega}}$ indexed by the set $\omega^{<\omega} = \bigcup_{n\in\omega}\omega^n$ and consisting of clopen sets ${\uparrow}\alpha = \{\beta \in \omega^\omega : \beta|n = \alpha\}$.
The following lemma completes the proof of the theorem. □

Lemma 1

The countable family $\{U_{\uparrow\alpha}\}_{\alpha\in\omega^{<\omega}}$ is an s*-network at x.

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Since $\{U_\alpha\}_{\alpha\in\omega^\omega}$ is a neighborhood base at x, there exists $\alpha \in \omega^\omega$ such that $U_\alpha \subset O_x$.

To finish the proof of Lemma 1, it suffices to prove another □

Lemma 2

There exists $k \in \omega$ such that the set $U_{\uparrow\alpha|k}$ contains infinitely many points x_n, $n \in \omega$.

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Without loss of generality we can assume that $x_n \neq x$ for all $n \in \omega$.
For every neighborhood $U \subset X$ of x consider the set $F(U) = \{n \in \omega : x_n \in U\} \subset \omega$. It follows that

$$\mathcal{F} := \{F(U) : U \text{ is a neighborhood of } x\}$$

is a free filter on ω and the family $\{F(U_\alpha)\}_{\alpha \in \omega^\omega}$ is a monotone base for $\mathcal{F}$.
We claim that the filter $\mathcal{F}$ is analytic (as a subspace of the power-set $\mathcal{P}(\omega)$, endowed with the natural compact metrizable topology). □

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A topological space X is called analytic if X is continuous image of ω^ω.

Known Fact

A metrizable separable space X is analytic if and only if X has a compact resolution, which is a family $(K_\alpha)_{\alpha\in\omega^\omega}$ of compact subsets of X such that $X = \bigcup_{\alpha\in\omega^\omega} K_\alpha$ and $K_\alpha \subset K_\beta$ for all $\alpha \le \beta$ in ω^ω.

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We apply this characterization to prove the analycity of the filter $\mathcal{F}$ on ω generated by the base consisting of the sets $F(U_\alpha) = \{n \in \omega : x_n \in U_\alpha\}$, $\alpha \in \omega^\omega$.

Observe that for every $\alpha \in \omega^\omega$ the set ${\uparrow}F(U_\alpha) := \{F \subset \omega : F(U_\alpha) \subset F\}$ is a compact subset of $\mathcal{F}$ and moreover $\mathcal{F} = \bigcup_{\alpha \in \omega^\omega} {\uparrow}F(U_\alpha)$, where ${\uparrow}F(U_\alpha) \subset {\uparrow}F(U_\beta)$ for any $\alpha \le \beta$ in ω^ω.

So, $({\uparrow}F(U_\alpha))_{\alpha \in \omega^\omega}$ is a compact resolution of the subspace $\mathcal{F} \subset \mathcal{P}(\omega)$ and hence the filter $\mathcal{F}$ is analytic and meager.

By the Talagrand's characterization of meager filters on ω, there exists a finite-to-one map $\varphi : \omega \to \omega$ such that for every $F \in \mathcal{F}$ the image $\varphi(F)$ has finite complement in ω.

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Observe that for every $\alpha \in \omega^\omega$ the set $\uparrow F(U_\alpha) := \{F \subset \omega : F(U_\alpha) \subset F\}$ is a compact subset of $\mathcal{F}$ and moreover $\mathcal{F} = \bigcup_{\alpha \in \omega^\omega} \uparrow F(U_\alpha)$, where $\uparrow F(U_\alpha) \subset \uparrow F(U_\beta)$ for any $\alpha \le \beta$ in ω^ω.

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Main Theorem

Thus we have proved

Theorem

If a topological space X has a neighborhood ω^ω-base at a point $x \in X$, then X has a countable s^*-network at x.

This theorem has many nice corollaries, for example:

Corollary (generalizing famous Arhangel'ski theorem)

Each countably tight Hausdorff Lindelöf space X with a neighborhood ω^ω-base at each point has cardinality $|X| \le \mathfrak{c}$.

This corollary is a consequence of Main Theorem and

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Each countably tight Hausdorff space X with countable s^*-character has cardinality $|X| \le 2^{L(X)}$ where $L(X)$ is the Lindelöf number of X.

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References

More information on ω^ω-bases and s*-networks can be found in the paper-book:

T.Banakh, *Topological spaces with an ω^ω-base*, 105 pages (http://arxiv.org/abs/1607.07978).

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