INTERPLAY BETWEEN TWO GENERALIZATIONS OF FIRST COUNTABILITY

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We shall discuss the interplay between two generalizations of the well-known

Definition 1. A topological space X is *first-countable* at a point $x \in X$ if there exists a countable neighborhood base $\{U_n\}_{n\in\omega}$ at x.

There are two directions of generalizing the first-countability: (i) to use more general index sets than ω and (ii) to look for countable networks instead of countable neighborhood bases.

The first approach leads to the following

Definition 2. Let $(P, \le)$ be a directed poset. A neighborhood base $\mathcal{B}$ at a point x of a topological space X is called a *P-base at x* if it admits a monotone enumeration $\mathcal{B} = \{U_p\}_{p\in P}$ (which means that $U_q \subset U_p$ for every $p \le q$).

We shall be especially interested in P-bases for the set ω^ω of all functions from ω to ω, endowed with the natural partial order. In the literature ω^ω-bases often are called *$\mathfrak{G}$-bases*.

The second approach leads to

Definition 3. Let $\mathcal{C}$ be a family of subsets of a topological space X. A family $\mathcal{N}$ of subsets of X is called a *$\mathcal{C}^*$-network* at a point $x \in X$ if for each neighborhood $O_x \subset X$ and each set $C \in \mathcal{C}$ accumulating at x there is a set $N \in \mathcal{N}$ such that $x \in N \subset O_x$ and the intersection $C \cap N$ is infinite.

For each topological space X we are interested in three concrete families:

- sc of all convergent sequences in X;
- s of all countable subsets in X;
- p of all subsets in X.

For any family $\mathcal{N}$ of subsets of a topological space X and any point $x \in X$ we have the implications:

$$\text{neighborhood base at } x \Rightarrow \mathsf{p}^*\text{-network at } x \Rightarrow \mathsf{s}^*\text{-network at } x \Rightarrow \mathsf{cs}^*\text{-network at } x.$$

Following Tsaban and Zdomskyy, we define a topological space X to have the *strong Pytkeev property* at a point $x \in X$ if it has a countable p^*-network at x.

The main result of the talk is the following (important) theorem proved in [1, §6.4].

Theorem 1. *If a topological space X has a neighborhood ω^ω-base at a point $x \in X$, then X has a countable s^*-network at x for the family s of all countable subsets of X. If X is countably tight at x, then X has the strong Pytkeev property at x.*

References

[1] T. Banakh, *Topological spaces with an ω^ω-base*, 105 pp. preprint (https://arxiv.org/abs/1607.07978).