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ON THE CONSISTENCY OF SOME PARTITION THEOREMS FOR CONTINUOUS COLORINGS, AND THE STRUCTURE OF $\aleph_1$-DENSE REAL ORDER TYPES

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We present some techniques in c.c.c. forcing, and apply them to prove consistency results concerning the isomorphism and embeddability relations on the family of $\aleph_1$-dense sets of real numbers. In this direction we continue the work of Baumgartner [2] who proved the axiom BA stating that every two $\aleph_1$-dense subsets of $\mathbb{R}$ are isomorphic, is consistent. We e.g. prove Con(BA+$(2^{\aleph_0}>\aleph_2)$). Let $\langle K^H, \leqslant\rangle$ be the set of order types of $\aleph_1$-dense homogeneous subsets of $\mathbb{R}$ with the relation of embeddability. We prove that for every finite model $\langle L, \leqslant\rangle$: Con(MA+ $\langle K^H, \leqslant\rangle \simeq \langle L, \leqslant\rangle$) iff L is a distributive lattice. We prove that it is consistent that the Magidor–Malitz language is not countably compact. We deal with the consistency of certain topological partition theorems. E.g. We prove that MA is consistent with the axiom OCA which says: "If X is a second countable space of power $\aleph_1$, and $\{U_0, \ldots, U_{n-1}\}$ is a cover of $D(X) \stackrel{\text{def}}{=} X \times X - \{\langle x, x\rangle \mid x \in X\}$ consisting of symmetric open sets, then X can be partitioned into $\{X_i \mid i \in \omega\}$ such that for every $i \in \omega$ there is $l < n$ such that $D(X_i) \subseteq U_l$". We also prove that MA + OCA $\Rightarrow 2^{\aleph_0} = \aleph_2$.

Introduction

The purpose of this paper is to prove consistency results about partitions of second countable spaces of power $\aleph_1$, and about the relations of embeddability and isomorphism between sets of real numbers of power $\aleph_1$.

Our intention is not only to prove new results, but also to present the techniques used. Because of this reason, in the first sections, we tried as much as possible to present applications in which the proofs were technically simple, and in which only one technique was being used at a time. Thus we sometimes had to repeat ourselves, and in one case we chose to reprove a theorem from [1], though in a different way. On the other hand we sometimes omit the proof of some details which resemble previous arguments.

The starting point of this paper is the theorem of Baumgartner [2] that *the axiom* BA, *which says that every two* $\aleph_1$*-dense sets of real numbers are order-isomorphic, is consistent.* Baumgartner in fact proved that MA+BA is consistent. The isomorphization of two $\aleph_1$-dense sets of real numbers was done by means of a c.c.c. forcing set. This suggested that maybe $\mathrm{MA}_{\aleph_1}$ already implies BA.

The negative answer to the above question was found by Shelah. He invented two techniques: the club method and the explicit contradiction method. Using these methods Shelah [1] proved that $\mathrm{MA}_{\aleph_1}$ was consistent with the existence of an entangled set (see Section 7), thus showing that $\mathrm{MA}_{\aleph_1} \not\Rightarrow \mathrm{BA}$.

Avraham [1] then found another way to refute BA. By means of the club method he constructured a universe V satisfying MA and a set of real numbers of power $\aleph_1$, $A \in V$, such that every 1-1 uncountable $g \subseteq A \times A$ contained an uncountable order preserving function. Such an A is not isomorphic to $A^* \overset{\mathrm{def}}{=} \{-a \mid a \in A\}$, thus $V \models \neg\mathrm{BA}$.

Answering a question of Avraham, Shelah [1] proved that it is consistent that every 1-1 $g \subseteq \mathbb{R} \times \mathbb{R}$ of power $\aleph_1$ can be represented as the union of countably many monotonic functions. The proof involved a new trick: The preassignment of colors (see Section 3).

The club method

The club method plays the most central role in this paper. We explain in what context one can try to use this method. Let $|A| = \aleph_1$, and let R be a binary relation on A. Suppose $R = \bigcup_{i \in \omega} B_i \times C_i$, (in this case we say that R has a countable semibase). By the club method one can try to construct a c.c.c. forcing set which adds to V an uncountable subset of A^n which has various homogeneity properties with respect to R. E.g. one might want to add an uncountable $g \subseteq A \times A$ such that for every $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle \in g$, $\langle a_1, a_2\rangle \in R$ iff $\langle b_1, b_2\rangle \in R$. (This is the case of adding an order preserving function.) Note that if X is a second countable space and $R \subseteq X \times X$ is open, then R has a countable semibase, hence $<_R$ and $_R>$ have countable semibases.

The club method makes the problem of isomorphizing two $\aleph_1$-dense subsets of $\mathbb{R}$ just one special option in a wide spectrum of possibilities.

In the beginning, we knew to apply the club method only when the ground model satisfied CH. After understanding the exact role of CH it was possible to replace it by an axiom denoted by A1 which may holds also in the absence of CH. A1 has the property that if $V \models \mathrm{A1}$ and P is a c.c.c. forcing set of power $<2^{\aleph_1}$, then V^P also satisfies A1. Hence one can carry out a finite support iteration of length $2^{\aleph_1}$ consisting of general c.c.c. forcing sets and 'club method' forcing sets. In this way we obtain the consistency of $\mathrm{BA} + (2^{\aleph_0} > \aleph_2)$ which could not have been obtained by the method of [2].

The other techniques described in this paper are easily combined with the club method in many different ways, thus yielding a rich variety of consistency results.

Summary of results

1. The club method and the semiopen coloring axiom

In this section we present the club method by means of an application. Let X denote a second countable space of power $\aleph_1$, let U be a symmetric open subset of $X \times X$, and for a set A let $D(A) = A \times A - \{\langle a, a \rangle \mid a \in A\}$. The semi open coloring axiom (SOCA) says: "For every X and U as above there is an uncountable $A \subseteq X$ such that either $D(A) \subseteq U$ or $D(A) \cap U = \emptyset$". In Section 1 we prove that MA+SOCA is consistent. This is probably the simplest application of the club method.

In addition we prove in Section 1 the consistency of a certain strengthening of SOCA, we prove some corollaries of SOCA, and bring some counter-examples.

2. The explicit contradiction method and the increasing set axiom

A set $S \subseteq \mathbb{R}$ of cardinality $\aleph_1$ is called an increasing set if for every $n \in \omega$ and a set $\{\langle a(\alpha, 0), \ldots, a(\alpha, n-1) \rangle \mid \alpha < \aleph_1\} \subseteq A^n$ of pairwise disjoint n-tuples there are $\alpha, \beta < \aleph_1$ such that for every $i < n$, $a(\alpha, i) < a(\beta, i)$.

Suppose $A \in V$ is increasing, and we want to construct a universe $W \supseteq V$ which satisfies MA and in which A retains its increasingness. The problem is that when we iterate c.c.c. forcing sets in order to take care of MA it may happen (and indeed it does happen if $V \models$CH) that some of the iterands P_i force that A is not increasing. The way in which this difficulty is overcome, is that we construct a c.c.c. forcing set Q such that $\Vdash_Q$ (P_i is not c.c.c) $\wedge$ (A is increasing). Hence forcing through Q retains the increasingness of A and frees us from forcing through P_i. The particular method in which this is done is called the explicit contradiction method.

Section 2 is devoted to the proof that $\mathrm{MA}_{\aleph_1}$ is consistent with the existence of an increasing set. Indeed $\mathrm{MA}_{\aleph_1} \Rightarrow A$ is increasing iff every uncountable 1-1 $g \subseteq A \times A$ contains an uncountable order preserving function. Thus what we prove in Section 2 coincides with Theorem 2 of [1]. However, since this is the simplest application of the explicit contradiction method, and since the proof we present can be used to retain also other properties of A, we take the liberty to reprove Theorem 2 of [1].

3. The open coloring axiom, and how to preassign colors

Let X denote a second countable space of power $\aleph_1$. An open cover $\mathcal{U} = \{U_0, \ldots, U_{n-1}\}$ of $D(X)$ consisting of symmetric sets is called an open coloring of

X. $A \subseteq X$ is $\mathcal{U}$-homogeneous if for some $i < n$, $D(A) \subseteq U_i$. Let OCA be the axiom: "For every X and $\mathcal{U}$ as above X can be partitioned into countably many $\mathcal{U}$-homogeneous subsets". Let ISA be the axiom: "There exists an increasing set".

Trying to strengthen SOCA, and Theorem 6 of [1], we prove that MA+SOCA+OCA+ISA is consistent. The new element in the proof is a use of the so-called preassignment of colors. Let X, $\mathcal{U}$ be as above, and let $A \subseteq \mathbb{R}$ be an increasing set. We want to partition X into countably many $\mathcal{U}$-homogeneous subsets without destroying the increasingness of A. There is a method to assign to each $a \in X$ a color $i(a) < n$ such that there is a c.c.c forcing set P which partitions X into countably many $\mathcal{U}$-homogeneous sets, in this partition every $a \in X$ belongs to a set with color $i(a)$, and P does not destroy increasingness of A. The preassignment of colors resembles Theorem 6 of [1], but here we have one additional trick devised in order to retain the increasingness of A.

OCA can be generalized to colorings of n-tuples rather than colorings of pairs. For $\nu, \xi \in {}^{\omega \geq}2$ let $\nu \wedge \xi$ denote the maximal common initial segment of ν and ξ. For $A \subseteq {}^{\omega}2$ let

$$T[A] = \{\nu \wedge \xi \mid \nu, \xi \in A \text{ and are distinct}\}.$$

For $\nu, \xi \in {}^{\omega >}2$ and $l = 0, 1$, let $\nu <_l \xi$ denote the fact that $\nu^\frown\langle l \rangle$ is an initial segment of ξ. Let σ, τ be finite subsets of ${}^{\omega >}2$. $\sigma \sim \tau$ means that $\langle \sigma, <_0, <_1 \rangle \cong \langle \tau, <_0, <_1 \rangle$.

Let TCAm be the axiom saying: "If $\langle C_0, \ldots, C_{k-1} \rangle$ is a partition of the unordered m-tuples of ${}^{\omega >}2$, and $A \subseteq {}^{\omega}2$ is of power $\aleph_1$, then there is a partition of A $\{A_i \mid i \in \omega\}$ such that for every $i \in \omega$ and two subsets σ_1, σ_2 of A_i of power $m+1$: if $T[\sigma_1] \sim T[\sigma_2]$, then there is $j < k$ such that $T[\sigma_1], T[\sigma_2] \subseteq C_j$".

In Section 3 we prove that Ma$+\bigwedge_{m \in \omega}TCAm$ is consistent. TCA1 is implied by OCA, and MA+TCA1 $\Rightarrow$ OCA.

TCAm has also a topological equivalent but its formulation is not very transparent. The more direct and stronger generalization of OCA remains open.

We conclude Section 3 with another axiom concerning partitions. Let X, Y be a second countable spaces such that $|X| = \aleph_1$, and Y does not contain isolated points; let f be a symmetric continuous function from $D(X)$ to Y. Let NWDA be the axiom which says: "If X, Y, f are as above, then there is a partition $\{A_i \mid i \in \omega\}$ of X such that for every $i, j \in \omega$ $f(A_i \times A_j - \{\langle a, a \rangle \mid a \in X\})$ is nowhere dense". We prove that MA+NWDA is consistent.

We did not investigate the relationship of NWDA with other axioms and its possible generalizations.

4. *The semi open coloring axiom does not imply the open coloring axiom; the tail method*

In Section 4 we prove that SOCA+MA+$(2^{\aleph_0} = \aleph_2) \not\Rightarrow$ OCA. Indeed, in Section 5 we prove that MA+SOCA is consistent with $2^{\aleph_0} > \aleph_2$, and in Section 11 we prove that $MA+2^{\aleph_0}>\aleph_2 \Rightarrow \neg OCA$, hence the result of Section 4 becomes less interesting. But the proof serves well in demonstrating an additional trick called the tail method. This trick is used also in Sections 9 and 10, but there, the technical details are somewhat more complicated.

5. *Enlarging the continuum beyond* $\aleph_2$

In Baumgartner's proof of the consistency of BA, the construction of a c.c.c. forcing set which isomorphizes $\aleph_1$-dense sets of real numbers, is done under the assumption of CH. So in the universe satisfying BA the continuum had to be $\aleph_2$. The substitute for CH in the application of the club method was found by Shelah. This immediately implied that BA is consistent with $2^{\aleph_0}>\aleph_2$. In this section we demonstrate this method by proving that $MA+SOCA+(2^{\aleph_0}>\aleph_2)$ is consistent.

6. MA, OCA *and the embeddability relation on* $\aleph_1$*-dense real order types*

Let $K=\{A\subseteq\mathbb{R} \mid A\neq\emptyset, A$ has no endpoints and every interval of A has cardinality $\aleph_1\}$. For $A, B\in K$ let $A\preccurlyeq B$ and $A\cong B$ respectively mean that $\langle A,<\rangle$ is embeddable or isomorphic to $\langle B,<\rangle$. Let $A\in K$. A is homogeneous if for every $a, b\in A$ there is an automorphism f of $\langle A,<\rangle$ such that $f(a)=b$. Let $K^H=\{A\in K \mid A$ is homogeneous$\}$. Let $N(A,B)$ mean that there is $C\in K$ such that $C\preccurlyeq A$ and $C\preccurlyeq B$; $A\perp B\equiv\neg N(A,B)$ and $A \perp\!\!\!\perp B\equiv A\perp B\wedge A\perp B^*$. Let NA be the axiom: $(\forall A,B\in K)\ N(A,B)$.

A great part of this work was motivated by questions about the possible structure of K and K^H. Since SOCA easily implies $(\forall A,B\in K)\ (N(A,B)\cup N(A,B^*))$ it was natural to ask whether it also implied NA. Since "A is increasing" implies $\neg N(A,A^*)$, this question was answered in Section 3. There was still another reason why MA+OCA+ISA was interesting. Shelah proved the consistency of the following axiom: "There are $A,B\in K^H$ such that: $A\cong A^*$, $B\cong B^*$, $A\perp\!\!\!\perp B$, $A\cup B\in K^H$ and for every $C\in K^H$ either $C\cong A$ or $C\cong B$ or $C\cong A\cup B$".

It was of interest to us to find whether in this axiom one can make the modification that $A\perp A^*$ and $A^*\cong B$. In Section 6 we indeed show that this modified axiom follows from MA+OCA+ISA.

In fact MA+OCA almost determines the structure of K^H and K. If MA+OCA is conjuncted with ISA, then K^H is as above, if MA+OCA is conjuncted with $\neg$ISA, then BA holds.

7. *Relationship with the weak continuum hypothesis*

The weak continuum hypothesis WCH is the statement that $2^{\aleph_0}<2^{\aleph_1}$. In Section 7 we first show that $BA\Rightarrow\neg WCH$. The question that naturally arises is what happens if BA is weakened and is replaced by NA. We prove that unlike BA, NA is consistent with WCH.

This automatically implies that NA$\not\Rightarrow$BA. The fact that MA+NA$\not\Rightarrow$BA follows from the results of Section 9.

In the proof of NA+WCH we introduce a forcing which makes two members of K near. This is a simple version of a forcing set which isomorphizes two members of K.

One can consider the following strengthening of NA. Let DNA be the following axiom: "If $A, B \in K$, then there is an uncountable order preserving function $g \subseteq A \times B$ such that Dom(g), Rng(g)$\in K$ and are dense in A and B respectively. Section 7 is concluded with a proof that NA$\Rightarrow$DNA.

8. A weak form of Martin's axiom, the consistency of the incompactness of the Magidor-Malitz quantifiers.

Let MML denote the Magidor–Malitz language. In [7] Magidor and Malitz proved that $\diamondsuit_{\aleph_1}\Rightarrow$"MML is countably compact". This suggested the following question: "Construct a universe in which MML is not countably compact". A first solution to this problem was found by Shelah (unpublished) using methods of Avraham. Shelah's solution involves properties of Suslin trees which are expressible by MML sentences. The result of Shelah is that the countable incompactness of MML is consistent with CH.

In Section 8 we bring a simpler solution to this question, here we obtain a universe in which MA+($\aleph_1 < 2^{\aleph_0}$)+(MML is not countably compact) holds.

Let $A \in K$ and $k \in \omega$, A is k-entangled if for every sequence $\{\langle a(\alpha, 0), \ldots, a(\alpha, k-1)\rangle \mid \alpha < \aleph_1\} \subseteq A^k$ of pairwise disjoint 1-1 sequences, and for every $\langle \varepsilon(0), \ldots, \varepsilon(k-1)\rangle \in \{0,1\}^k$ there are $\alpha_0, \alpha_1 < \aleph_1$ such that for every $i<k$, $a(\alpha_{\varepsilon(i)}, i) < a(\alpha_{1-\varepsilon(i)}, i)$. The k-entangleness of A can be expressed by an MML sentence and $\text{MA}_{\aleph_1} \Rightarrow \neg(\exists A \in K)(\forall k \in \omega)$ (A is k-entangled). Let $W \models \text{MA}_{\aleph_1} + (\forall k \in \omega)(\exists A \in K)$ (A is k-entangled). Hence in W MML is not countably compact.

The notion of entangledness was defined by Shelah in [1]. There, it is proved that for every $k \in \omega$, $\text{MA}_{\aleph_1} + (\exists A \in K)$ (A is k-entangled) is consistent. It is somewhat more complicated to prove that $\text{MA}_{\aleph_1} + (\forall k \in \omega)(\exists A \in K)$ (A is k-entangled) is consistent. We prove this fact in Section 8.

The other question considered in Section 8 is whether iterating forcing sets obtained by means of the club method can yield a universe satisfying $\text{MA}_{\aleph_1}$. To prove that this is not so we define a property, denoted by s.c.c., and stronger than the countable chain condition, such that every forcing set gotten from the club method has this property. On the other hand we prove that a finite support iteration of s.c.c. forcing sets does not destroy Suslin trees. We hence obtain that OCA, SOCA, NA, etc. are consistent with the existence of a Suslin tree.

9. The isomorphizing forcing, and more on the possible structure of K

In this section we first construct for $A, B \in K$ a c.c.c. forcing set P such that $\Vdash_P A \cong B$. This construction is a basic tool for results concerning the possible structure of K. This construction can be carried out under assumptions weaker than CH, hence we can prove that BA is consistent with $2^{\aleph_0} > \aleph_2$. The other important property of this construction is that it enables to isomorphize two sets leaving some other sets far. E.g., we prove that if $A, B, C \perp\!\!\!\perp D$, then there is a c.c.c. P which isomorphizes A and B and keeps $C \perp\!\!\!\perp D$.

$A \in K$ is rigid if $\langle A, <\rangle$ has no automorphisms other than the identity. Let

$$\text{RHA} \equiv (\forall A \in K)\,(\exists B, C \in K)\,((B, C \subseteq A) \wedge (B \text{ is rigid}) \wedge (C \text{ is homogeneous})).$$

Note that $\text{RHA} \Rightarrow \neg\text{CH}$.

Combining the construction of isomorphizing forcing sets with the explicit contradiction method and the tail method we prove the consistency of MA+ RHA.

10. *The structure of K and K^H when K^H is finite*

In Section 6 we prove that $\text{MA}_{\aleph_1} \Rightarrow K^H/\cong$ is partially ordered by $\leqslant$. Clearly $*$ is an automorphism of $\langle K^H/\cong, \leqslant\rangle$. Let $K^{HZ} = (K^H/\cong) \cup \{\emptyset\}$, $\langle K^{HZ}, \leqslant, *\rangle$ is a partially ordered set with an involution. In Section 10 we prove the following theorem: Let $\langle L, \leqslant, *\rangle$ be a finite partially ordered set with an involution: Then MA+ $(K^{HZ} \cong L)$ is consistent iff L is a finite distributive lattice with an involution.

This theorem was preceded by the following result by Shelah: It is consistent that $K^{HZ} = \{0, a, b, c\}$ where $a \wedge b = 0$, $a^* = a$, $b^* = b$ and $c = a \vee b$. Avraham and Rubin then showed (Section 3, 6) that K^{HZ} may by $\{0, a, b, c\}$ where $a \wedge b = 0$, $a = b^*$ and $c = a \vee b$.

Some results in the same direction were proved by Rubin for the class $K^{IIH} \overset{\text{def}}{=} \{A \in K^H \mid A \text{ is of the second category}\}$.

We also prove in Section 10 some results about the possible infinite K^{HZ}'s.

11. MA+OCA *implies* $2^{\aleph_0} = \aleph_2$.

Until the writing of this paper had been almost finished, we believed that the method to enlarge $2^{\aleph_0}$ beyond $\aleph_2$ worked for all applications of the club method. We realized that CH was used not only in the application of the club method, but also in order to e.g., get from $A, B \in K$ $A', B' \in K$ such that $A' \subseteq A$, $B' \subseteq B$ and $A' \perp\!\!\!\perp B'$. However this could be done too without assuming CH. Finally we noticed that, indeed, we did not know to preassign colors (Section 3) without CH, and we did not know how to prove the consistency of SOCA1 (Section 1) and the results of Section 10 without assuming CH in the intermediate models.

Shelah then found that at least in the case of OCA, the failure to prove the consistency of OCA+MA+$(2^{\aleph_0} > \aleph_2)$ followed from the fact that this axiom was false. He found a c.c.c. forcing set P of power $\aleph_2$, and $\aleph_2$ dense subsets of P, such that if V contains a filter of P which intersects all these dense sets, then V contains an open coloring of a set $A \subseteq {}^{\omega}2$ of power $\aleph_1$ for which there is no partition of A into countably many homogeneous sets. Section 11 contains this result.

Whether the results of Section 10 and SOCA1 are consistent with MA+ $(2^{\aleph_0}>\aleph_2)$ remains open.

Main open problems

In the paper we mention many open problems, they appear in the relevant context. Let us mention here those problems which, we believe, require new techniques.

(1) (Baumgartner) Is it consistent that every two $\aleph_2$-dense sets are isomorphic? More generally, are the axioms appearing in this paper consistent when we replace $\aleph_1$ by $\aleph_2$?

(2) The axioms mentioned in this paper are all consistent with MA. We do not know how to prove the consistency of similar axioms which contradict MA. E.g., is the following axiom consistent: $\neg$BA$+(\forall A, B \in K)\,(A \leqslant B)$? Is the following axiom consistent: OCA$+2^{\aleph_0}>\aleph_2$?

(3) Let OCA(m, k) be the following axiom: "For every second countable space X of power $\aleph_1$ and every finite open cover $\mathcal{U}$ of X^m, there is a partition $\{X_i \mid i \in \omega\}$ of X such that for every $i \in \omega$, X_i^m intersects at most k members of $\mathcal{U}$." Does there exist a k for which OCA(m, k) is consistent? In fact we do not know the answer even for $m = 3$, and even if the axiom is weakened to require only the existence of one uncountable subset A of X such that A^m intersects at most k members of $\mathcal{U}$.

(4) Are some of the axioms mentioned consistent with the existence of a second category subset of $\mathbb{R}$ of power $\aleph_1$? E.g. are NA$+(\exists A \in K)$ (A is of the second category) and SOCA$+(\exists A \in K)$ (A is of the second category) consistent?

Historical remarks

The club method, explicit contradiction method, the method to enlarge $2^{\aleph_0}$ beyond $\aleph_2$ are due to Shelah. The tail method is due to Rubin. The method of preassigning colors is due to Shelah, but an additional trick was added by Avraham and Rubin. Section 1 dealing with SOCA is mainly the work of Avraham and Rubin. Section 2 is another proof of a theorem by Avraham and Shelah in [1]. The axiom OCA appearing in Section 3 and its corollaries concerning the structure of K appearing in Section 6 are due to Avraham and Rubin. The axiom TCAm which generalizes OCA is due to Shelah and the axiom NWDA is due to Rubin. The proof that SOCA$\not\Rightarrow$OCA appearing in Section 4 is due to Rubin. Section 5 dealing with how to enlarge $2^{\aleph_0}$ beyond $\aleph_2$ is due to Shelah. Section 7 dealing with the relationship with WCH is due to Shelah. The weak Martin's axiom appearing in Section 8 and the proof that it is consistent with the existence of Suslin trees is due to Avraham and Rubin. The proof that MML may be countably incompact is due to Rubin. This theorem was first proved by Shelah using other methods. The proof was a slight improvement of a theorem of Shelah in [1].

The isomorphizing forcing in Section 9 is due to Shelah. BA1 as well as RHA are due to Rubin. RHA uses the tail method as well as an important lemma essentially due to Shelah. This lemma states that if $A, B \perp\!\!\!\perp C, D$ then it is possible to isomorphize A and B keeping $C \perp\!\!\!\perp D$. Section 10 which deals with the structure of K and K^H when K^H is finite is due to Rubin. The theorem stating that $MA+OCA \Rightarrow 2^{\aleph_1}=\aleph_2$ appearing in Section 11 is due to Shelah.

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For the reader's convenience we include here an index of axioms and some notations used in this work.

1. The club method and the semiopen coloring axiom

In this section we present the club method which is the main technique in this paper. We prove a theorem in which the club method is used. This theorem is perhaps the simplest application of this method.

For a set A let $D(A)=A\times A-\{\langle a, a\rangle \mid a \in A\}$. A function f from the set of unordered pairs of a set A to $\{0, 1\}$ is called a coloring of A in two colors. We regard f as a symmetric function from $D(A)$ to $\{0, 1\}$. A subset $B \subseteq A$ is called f-homogeneous, or in short homogeneous, if $f \restriction D(B)$ is a constant function; we say that B is of color l, or in short B is l-colored if the value of $f \restriction D(B)$ is l.

From now on X denotes a second countable topological Hausdorff space of cardinality $\aleph_1$. Let f be a coloring of X in two colors, f is called a semiopen coloring (SOC), if $f^{-1}(1)$ is open in $X \times X$.

Let the semiopen coloring axiom be the following axiom.

Axiom SOCA. For every X and a SOC f of X, X contains an uncountable f-homogeneous subset.

Theorem 1.1. SOCA *is consistent with* ZFC.

Proof. We prove the following claim. Let $V \models \mathrm{CH}$, and let f be a SOC of X such that X has no uncountable homogeneous subset of color 0; then there is a c.c.c. forcing set of power $\aleph_1$, $P = P_{X,f}$, such that in V^P, X contains an uncountable homogeneous set of color 1.

By the method of Solovey and Tenenbaum [9], this claim suffices in order to prove the theorem. More specifically we start with a universe satisfying CH+ $(2^{\aleph_1} = \aleph_2)$ and carry out an iteration with direct limits $\{P_\alpha \mid \alpha < \aleph_2\}$, in which each $(\alpha+1)$st iterand is the P_α-name of some forcing set of the form $P_{X,f}$.

We thus turn to the construction of $P_{X,f}$ assuming that CH holds, and X and f are given. We first need a model of the form $\langle \aleph_1, <, \ldots, \rangle$ that includes the information about X and f, and that encompasses enough set theory. In order not to repeat the same definition over and over, we shall at this point fix a model that will serve us also in the future. Let $H(\aleph_1)$ be the set of hereditarily countable sets. By CH, $|H(\aleph_1)| = \aleph_1$. We choose a 1-1 correspondence h between $H(\aleph_1)$ and $\aleph_1$. Let $M^0 = \langle \aleph_1, <, h, \in_1 \rangle$ where $\alpha \in_1 \beta$ iff $h(\alpha) \in h(\beta)$. In order not to have two belonging relation symbols we shall denote $\in_1$ by $\in$ and will refrain from using "$\alpha \in \beta$" to mean the usual belonging relation between countable ordinals; instead we shall write "$\alpha < \beta$". We reserve M^0 to mean the above model throughout this paper.

W.l.o.g. $X \subseteq \aleph_1$. Let $M = \langle M^0, X, f, T \rangle$; by this we mean that we expand M^0 by adding to it a unary predicate to represent X, a binary function symbol to represent f, and some binary relation symbol to represent some fixed countable base for X. T can be defined in the following way: let $\{U_i \mid i \in \omega\} \stackrel{\text{def}}{=} \mathcal{U}$ be a countable base for X; $T = \{\langle i, \alpha \rangle \mid i \in \omega \text{ and } \alpha \in U_i\}$.

For $\alpha < \aleph_1$, let M_α denote the submodel of M whose universe is α. Let $C_M = \{\alpha \mid M_\alpha \prec M\}$. C_M is a closed unbounded set (club).

A subset $A \subseteq \aleph_1$ is called C_M-separated or in short separated, if for every $\alpha, \beta \in A$ such that $\alpha < \beta$ there are $\gamma_1, \gamma_2 \in C_M$ such that $\gamma_1 < \alpha < \gamma_2 < \beta$.

Let λ be a cardinal and A be a set; we denote $P_\lambda(A) = \{B \subseteq A \mid |B| < \lambda\}$. Let $P_{X,f} = \{\sigma \in P_{\aleph_0}(X) \mid \sigma \text{ is homogeneous of color 1 and } \sigma \text{ is separated}\}$. The partial ordering on $P_{X,f}$ is set inclusion.

We show that $P_{X,f}$ is c.c.c. Suppose by contradiction that it is not; then it is easy to see that there is $\Gamma_1 = \{\sigma^i \mid i < \aleph_1\} \subseteq P_{X,f}$ such that:

(1) for every $i < j < \aleph_1$, $\sigma^i \cup \sigma^j$ is not homogeneous of color 1;

(2) for every $i < j < \aleph_1$: $|\sigma^i| = |\sigma^j|$, and $\sigma^i \cup \sigma^j$ is C_M-separated.

Let $\{\alpha_1^i, \ldots, \alpha_n^i\}$ be an enumeration of σ^i in an increasing order. Since σ^i is homogeneous of color 1 and since f is a SOC, there are $U_1^i, \ldots, U_n^i \in \mathcal{U}$ such that for every $k \neq l$, $\sigma_k^i \in U_k^i$ and $f(U_k^i \times U_l^i) = \{1\}$. Let Γ be an uncountable subset of Γ_1 such that for every $i, j \in \Gamma$ and for every $1 \leq k \leq n$, $U_k^i = U_k^j \overset{\text{def}}{=} U_k$. By reindexing we can assume that $\Gamma = \{\sigma^i \mid i < \aleph_1\}$. We thus conclude

(*) For every $i, j < \aleph_1$ and $1 \leq k \neq l \leq n$, $f(\alpha_k^i, \alpha_l^j) = 1$.

The next step which we call 'the duplication argument' is one of the central arguments in this paper. For a subset A of a topological space X, let $\mathrm{cl}(A)$ be the topological closure of A in X.

$\Gamma \subseteq X^n$ and X^n is second countable, hence for some countable $\Gamma_0 \subseteq \Gamma$, $\mathrm{cl}(\Gamma_0) = \mathrm{cl}(\Gamma)$. Let $\gamma \in C_M$ be such that $\Gamma_0 \in |M_\gamma|$. (More precisely we mean that $h(\Gamma_0) < \gamma$, but we shall always make this abuse of notation.) Note also that $\mathcal{U} \subseteq M_\alpha$ for every $\alpha \in C_M$. There is a formula in the language of M_γ and with the parameter Γ_0, $\varphi(x_1, \ldots, x_n)$, which says that $\langle x_1, \ldots, x_n \rangle \in \mathrm{cl}(\Gamma_0)$. Let $i < \aleph_1$ be such that $\gamma < \alpha_1^i$. We want to define by a downward induction a sequence of certain formulas $\varphi_l(x_1, \ldots, x_l)$, $l = 0, \ldots, n$, where $\varphi_n = \varphi$ and where $M \models \varphi_l[\alpha_1^i, \ldots, \alpha_l^i]$. For the sake of clarity we first show how to get φ_{n-1}. Let $\delta \in C_M$ and $\alpha_{n-1}^i < \delta < \alpha_n^i$. For every $\alpha \in |M_\delta|$, $M \models \psi[\alpha_1^i, \ldots, \alpha_{n-1}^i, \alpha]$ where $\psi(x_1, \ldots, x_{n-1}, x) \equiv (\exists x_n > x)\, \varphi(x_1, \ldots, x_n)$; for one can take x_n to be α_n^i. Since $M_\delta \prec M$, $M_\delta \models \psi[\alpha_1^i, \ldots, \alpha_{n-1}^i, \alpha]$. Hence $M_\delta \models \forall x\, \psi[\alpha_1^i, \ldots, \alpha_{n-1}^i, x]$, hence M satisfies the same formula. This means that $L \overset{\text{def}}{=} \{\beta \mid \langle \alpha_1^i, \ldots, \alpha_{n-1}^i, \beta \rangle \in \mathrm{cl}(\Gamma_0)\}$ is unbounded and thus uncountable. We assumed that X did not contain uncountable homogeneous sets of color 0, thus there are $\beta_1, \beta_2 \in L$ such that $f(\beta_1, \beta_2) = 1$. Let $U_1^n, U_2^n \in \mathcal{U}$ be disjoint sets such that $\beta_l \in U_l$ and $f(U_1^n \times U_2^n) = \{1\}$. Let

$$\varphi_{n-1}(x_1, \ldots, x_{n-1}) \equiv \exists x_1^n \exists x_2^n \left(\bigwedge_{l=1}^{2} (x_l^n \in U_l^n \wedge \varphi(x_1, \ldots, x_{n-1}, x_l^n)) \right).$$

Clearly $M \models \varphi_{n-1}[\alpha_1^i, \ldots, \alpha_{n-1}^i]$. Suppose φ_m has been defined and $M \models \varphi_m[\alpha_1^i, \ldots, \alpha_m^i]$. Repeating the same argument as before, there are disjoint $U_1^m, U_2^m \in \mathcal{U}$ and $\beta_l \in U_l^m$, $l = 1, 2$, such that $f(U_1^m \times U_2^m) = \{1\}$, and $M \models \varphi_m[\alpha_1^i, \ldots, \alpha_{m-1}^i, \beta_l]$. Let

$$\varphi_{m-1} = \exists x_1^m \exists x_2^m \left(\bigwedge_{l=1}^{2} (x_l^m \in U_l^m \wedge \varphi_m(x_1, \ldots, x_{m-1}, x_l^m)) \right).$$

Now we start with φ_0 and inductively pick β_l^j, $l = 1, 2$, $j = 1, \ldots, n$. Since $M \models \varphi_0$ there are $\beta_l^1 \in U_l^1$, $l = 1, 2$, such that $M \models \varphi_1[\beta_l^1]$. Suppose $\beta_l^1, \ldots, \beta_l^m$, $l = 1, 2$, were defined so that $\beta_l^j \in U_l^j$ and $M \models \varphi_m[\beta_l^1, \ldots, \beta_l^m]$, $l = 1, 2$. Hence β_l^{m+1} can be chosen to satisfy the same induction hypotheses. The fact that $M \models \varphi_n[\beta_l^1, \ldots, \beta_l^n]$ means that $\langle \beta_l^1, \ldots, \beta_l^n \rangle \in \mathrm{cl}(\Gamma_0)$. Since U_l^m is a neighborhood of β_l^m, there are $\boldsymbol{\alpha}_l \in \Gamma_0 \cap U_l^1 \times \cdots \times U_l^m$. By (*) and the choice of the U_l^j's, $\boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2$ is homogeneous of color 1, a contradiction. We have thus proved that $P_{X,f}$ is c.c.c.

The union of all elements of a generic subset of $P_{X,f}$ is a homogeneous subset of X of color 1. It remains to show that this union is indeed uncountable.

It suffices to show that for every $\sigma \in P_{X,f}$, $\{\alpha \mid \sigma \cup \{\alpha\} \in P_{X,f}\}$ is unbounded. Suppose by contradiction $\sigma = \{\alpha_1, \ldots, \alpha_n\}$ is a counterexample to this claim. Let $Qx\,\varphi(x)$ mean: "there are unboundedly many x's satisfying φ". Using the fact that there is $\delta \in C_M$ such that $\alpha_{n-1} < \delta < \alpha_n$, it is easy to see that $M \models Qx\,\varphi[\alpha_1, \ldots, \alpha_{n-1}, x]$ where $\varphi(x_1, \ldots, x_n) \equiv (\{x_1, \ldots, x_n\}$ is homogenesous of color $1) \wedge (\{y \mid \{x_1, \ldots, x_n, y\}$ is homogeneous of color $1\}$ is bounded). For every β satisfying $\varphi(\alpha_1, \ldots, \alpha_{n-1}, x)$ let ν_β be a bound as assured by φ. Let $\{\beta_i \mid i < \aleph_1\}$ be a separated set such that for every $i < j < \aleph_1$, $M \models \varphi[\alpha_1, \ldots, \alpha_{n-1}, \beta_i]$ and $\beta_j > \nu_{\beta_i}$. The set $\{\{\alpha_1, \ldots, \alpha_{n-1}, \beta_i\} \mid 0 < i < \aleph_1\}$ is an uncountable antichain in $P_{X,f}$, a contradiction. □

The use of topological terminology and especially the use of the Hausdorff condition in Theorem 1.1 was redundant; we did not lose however any generality. We now give an equivalent formulation of the theorem that does not involve topology. Let $|A| = \aleph_1$ and f be a coloring of A in two colors. A *semibase* for f is a family $\{\langle C_i, D_i\rangle \mid i < \alpha\}$ such that $f^{-1}(1) = \bigcup_{i<\alpha} C_i \times D_i$.

Theorem. *It is consistent with* ZFC, *that for every coloring f of $\aleph_1$ in two colors which has a countable semibase, $\aleph_1$ contains an uncountable f-homogeneous subset.*

Theorem 1.2 (Consequences of SOCA). *Assume* SOCA, *then:*

(a) *If $f \subseteq \mathbb{R} \times \mathbb{R}$ is a* 1-1 *uncountable function, then there is a monotonic uncountable $g \subseteq f$.* (A model satisfying this property was built in [1].)

(b) *If $f \subseteq \mathbb{R} \times \mathbb{R}$ is a* 1-1 *uncountable function, then there is an uncountable $g \subseteq f$ such that g or g^{-1} is a Lipschiz function.*

(c) *If $A \subseteq P(\omega)$ is uncountable, then either A contains an uncountable chain, or A contains an uncountable set of pairwise incomparable elements. If B is an uncountable Boolean algebra, then B contains an uncountable set of pairwise incomparable elements.* (A model satisfying this axiom was built by Baumgartner in [3].)

(d) *Let $R \subseteq D(X)$ be open, then there is an uncountable $A \subseteq X$ such that either $D(A) \subseteq R$, or $D(A) \cap R = \emptyset$ or $R \restriction A$ is a linear ordering on A.*

Proof. (a) Since $f \subseteq \mathbb{R} \times \mathbb{R}$, f is equipped with a second countable topology. Let c be the following coloring of $f: c(\boldsymbol{a}_1, \boldsymbol{a}_2) = 0$ if $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ is an order preserving function, and otherwise $c(\boldsymbol{a}_1, \boldsymbol{a}_2) = 1$. Since f is 1-1 both $c^{-1}(0)$ and $c^{-1}(1)$ are open, hence the claim of (a) follows.

(b) We regard f as a topological subspace of $\mathbb{R} \times \mathbb{R}$. For $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle \in f$ let $c(\langle a_1, b_1\rangle, \langle a_2, b_2\rangle) = 1$ if $|(b_2 - b_1)/(a_2 - a_1)| < 1$, otherwise $c(\langle a_1, b_1\rangle, \langle a_2, b_2\rangle) = 0$. Clearly c is a SOC, hence (b) follows.

(c) The relation of incomparability on $P(\omega)$ has a countable semibase since $\tau, \sigma \in P(\omega)$ are incomparable iff for some distinct $n, m \in \omega$, $n \in \tau \not\ni m$ and $n \notin \sigma \ni m$. Hence the first part of (c) follows.

Let B be an uncountable Boolean algebra. If B does not contain a countable dense subset, then by a theorem of Baumgartner [3], B contains an uncountable set of pairwise incomparable elements. Hence w.l.o.g. B contains a countable dense subset, so B is embeddable in $p(\omega)$. By the first part of (c), B contains a chain or an anti-chain. If the latter happens, then our claim is true; otherwise let C be an uncountable chain in B. A subset of $P(\omega)$ which is a chain must be embeddable in $\langle\mathbb{R}, <\rangle$, since the lexicographic ordering between the characteristic functions of the elements of C is identical with the containment relation on C, and on the other hand $p(\omega)$ together with its lexicographic order is isomorphic to a Cantor set.

Let $d \in C$ be such that $C^1 \overset{\text{def}}{=} \{c \in C \mid c \subseteq d\}$ and $C^2 \overset{\text{def}}{=} \{c \in C \mid d \subseteq c\}$ are uncountable. Let $f: C^1 \to C^2$ be a 1-1 function. By (a) there is an uncountable monotonic $g \subseteq f$. If g is order reversing let $D = \{c \cup (g(c) - d) \mid c \in \mathrm{Dom}(g)\}$, then D is an uncountable set of pairwise incomparable elements. If g is order preserving let $D = \{(d - c) \cup (g(c) - d) \mid c \in \mathrm{Dom}(g)\}$; again, D is as required.

(d) Let $R'(x, y) \equiv R(y, x)$, hence R' is open in $X \times X$. Let $f(x, y) = 1$ if $R(x, y)$ or $R'(x, y)$ holds; and otherwise $f(x, y) = 0$. Hence f is a SOC. Let A be an uncountable f-homogeneous subset of X. If A has color 0, then $D(A) \cap R = \emptyset$, and thus A is as required. Otherwise, for every distinct $a, b \in A$, $R(a, b)$ or $R'(a, b)$ holds. Let $g: D(A) \to \{0, 1\}$ be defined as follows: $g(a, b) = 1$ if $R(a, b)$ and $R'(a, b)$ hold; and otherwise $g(a, b) = 0$. g is a SOC, hence let B be an uncountable g-homogeneous subset of A. If the color of B is 1, then $D(B) \subseteq R$, hence B is as required; otherwise $R \restriction B$ is an antisymmetric connected relation on B. Let $<$ be a linear ordering of B such that $\langle B, <\rangle$ is embeddable in $\langle\mathbb{R}, <\rangle$. Let $c: D(B) \to \{0, 1\}$ be defined as follows: $c(a, b) = 1$ iff $a < b \Leftrightarrow R(a, b)$. Obviously c has a countable open semibase, and R is a linear ordering on any c-homogeneous set. □

Strengthenings of SOCA

Proposition 1.3. SOCA + MA *is consistent with* ZFC.

Proof. In the proof of the consistency of SOCA we iterated c.c.c. forcing sets. We had the freedom to include in the iteration any c.c.c iterands, and SOCA would have still held. So we interlace in the iteration all $P_{X,f}$'s and all c.c.c. forcing sets of power $\aleph_1$. If P is the forcing set gotten as the limit of such an iteration, then $V^P \vDash \mathrm{SOCA} + \mathrm{MA}$. □

Proposition 1.4. *Suppose* $V \vDash \mathrm{SOCA} + \mathrm{MA}$. *Let f be a* SOC *of a space X such that X does not contain uncountable* 0*-colored sets, then X is a countable union of* 1*-colored sets.*

Proof. Let P be the following forcing set.

$$P = \{f \mid \mathrm{Dom}(f) \in P_{\aleph_0}(X),\ \mathrm{Rng}(f) \subseteq \omega, \text{ and for every } i \in \omega,$$
$$f^{-1}(i) \text{ is a homogeneous set of color } 1\}.$$

It suffices to show that P is c.c.c. Let $\{f_\alpha \mid \alpha < \aleph_1\} \subseteq P$. W.l.o.g. for every $\alpha \neq \beta$, $\mathrm{Dom}(f_\alpha) \cap \mathrm{Dom}(f_\beta) = \emptyset$, and $\langle a_\alpha(0,0), \ldots, a_\alpha(0, m_0), \ldots, a_\alpha(n, 0), \ldots, a_\alpha(n, m_n)\rangle$ is a 1-1 enumeration of $\mathrm{Dom}(f_\alpha)$ such that for every $i = 0, \ldots, n$ and $j = 0, \ldots, m_i$, $f_\alpha(a_\alpha(i, j)) = 1$. We can further assume that for every $i = 0, \ldots, n$, $0 \leq j < k \leq m_i$ and $\alpha, \beta < \aleph_1$, $f(a_\alpha(i, j), a_\beta(i, k)) = 1$. Recalling that X does not contain uncountable 0-colored sets, we apply successively SOCA to the subsets $\{a_\alpha(i, j) \mid \alpha < \aleph_1\}$ of X. Hence we obtain an uncountable subset $A \subseteq \aleph_1$ such that for every distinct $\alpha, \beta \in A$ and for every i and j, $f(a_\alpha(i, j), \alpha_\beta(i, j)) = 1$. Hence every finite subset of $\{f_\alpha \mid \alpha \in A\}$ is compatible. □

Remark. Note that we needed a rather weak form of MA since P has the property that every uncountable subset of P contains an uncountable set of finitely compatible elements.

We do not know whether the analogue of Proposition 1.4 for the color 0 is true.

Question. Is conjunction of the following axioms consistent? MA+SOCA+ "There is a pair $\langle X, f\rangle$ such that f is a SOC of X, X does not contain uncountable 1-colored sets but X is not a countable union of 0-colored sets".

We can still say something about the analogue of 1.4. Let SOCA1 be the axiom which says that for every pair $\langle X, f\rangle$ such that f is a SOC of X: X contains an uncountable homogeneous set, and if for some $l \in \{0, 1\}$, X does not contain uncountable l-colored sets, then X is a countable union of $(1-l)$-colored sets.

Theorem 1.5. MA+SOCA1 *is consistent.*

Proof. The proof is as the proof of Theorem 1.1 except that the first claim in Theorem 1.1 has to be strengthened as follows.

Claim (CH). *Let f be a SOC of X, and X is not a countable union of 0-colored sets. Then there is a c.c.c forcing set $P^1_{X,f} = P$ of power $\aleph_1$ such that $\Vdash_P$ "X contains a 1-colored uncountable set".*

Proof. Assume X is not a countable union of 0-colored sets. Let $\{F_i \mid i < \aleph_1\}$ be an enumeration of all 0-colored closed subsets of X. Choose by induction a sequence $\{x_i \mid i < \aleph_1\} \subseteq X$ such that for every i, $x_i \notin \bigcup_{j \leq i} F_j \cup \{x_j \mid j < i\}$; this choice is possible since X is not a countable union of 0-colored sets. Let $Y = \{x_i \mid i < \aleph_1\}$: clearly Y is a second countable Hausdorff space of power $\aleph_1$ and $f \restriction Y$ is a SOC of Y. We show that Y does not contain a 0-colored uncountable subset. Suppoe it did, and let A be such an example. $\mathrm{cl}(A)$ is also homogeneous of color 0, hence for some $i < \aleph_1$, $\mathrm{cl}(A) = F_i$. Since A is uncountable, for some $j > i$, $A \ni x_j$. This contradicts the definition of $\{x_j \mid j < \aleph_1\}$.

Let $P^1_{X,f} = P_{Y, f\restriction Y}$, clearly $P^1_{X,f}$ is as desired. □

To prove Theorem 1.5, we start with a universe V satisfying CH $+ (2^{\aleph_1} = \aleph_2)$. We make a list of tasks which includes all possible names of pairs $\langle X, f \rangle$ and all possible names of c.c.c forcing sets of power $\aleph_1$. Let this list be $\{R_\alpha \mid \alpha < \aleph_2\}$. We define $\{P_\alpha \mid \alpha \leqslant \aleph_2\}$ as follows: P_0 is a trivial forcing set, and for limit δ $P_\delta = \bigcup_{\alpha<\delta} P_\alpha$. Suppose P_α has been defined. If R_α is a P_α-name of a c.c.c. forcing set we define $P_{\alpha+1} = P_\alpha * R_\alpha$. If R_α is a name of a pair $\langle X, f \rangle$ such that X is not a countable union of 0-colored sets, then $P_{\alpha+1} = P_\alpha * P^1_{X,f}$. In all other cases $P_{\alpha+1} = P_\alpha$. This concludes the proof of 1.5. □

Some easy counter-examples

One can try to strengthen SOCA in various ways.

(1) Increase the number of colors, namely consider f's from X to ω in which for every $i \in \omega$, $f^{-1}(i)$ is open.

(2) Consider colorings of unordered n-tuples rather than coloring of pairs.

(3) Consider colorings f in which for every i, $f^{-1}(i)$ is a Borel set.

(4) Try to decompose X into countably many homogeneous sets.

Appropriate versions of (2) and (4) are consistent, this will be proved in Section 3. (1) and (3) are inconsistent. We give counter-examples to (1)–(4).

Example 1.6. There is an open coloring f of the unordered pairs of ${}^{\omega}2$ in $\aleph_0$ colors, such that ${}^{\omega}2$ does not contain an uncountable homogeneous subset.

For distinct $\eta, \nu \in {}^{\omega}2$ let $f(\eta, \nu)$ be the maximal common segment of η and ν.

Example 1.7 (Blass [4]). There is an open coloring f of the unordered triples from ${}^{\omega}2$ in 2 colors such that ${}^{\omega}2$ does not contain an uncountable homogeneous subset.

Let $\eta, \nu, \xi \in {}^{\omega}2$ be distinct and $\eta < \nu < \xi$ lexicographically ordered. $f(\eta, \nu, \xi) = 0$ if the maximal common initial segment of ξ and η is a proper initial segment of the maximal common initial segment of ν and η. Otherwise $f(\eta, \nu, \xi) = 1$.

Example 1.8. There is $X \subseteq \mathbb{R} \times \mathbb{R}$ of power $\aleph_1$ and a SOC f of X such that X is not the countable union of homogeneous subsets.

Let $A \subseteq \mathbb{R}$ be a power $\aleph_1$, $X = A \times A$ and $f(\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle) = 1$ iff $\{\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle\}$ is a strictly order preserving function, and otherwise the value of f is 0.

Clearly f is a SOC of X. For $B \subseteq X$ let $D_B = \{a \in A \mid$ there are distinct $b_1, b_2 \in A$ such that $\langle a, b_1 \rangle, \langle a, b_2 \rangle \in B\}$. If B is a 1-colored homogeneous set, then $D_B = \emptyset$. If B is 0-colored, then it is easily seen that $|D_B| \leqslant \aleph_0$. Let $\{B_i \mid i \in \omega\}$ be a family of homogeneous subsets of X, and let $a \in A - \bigcup_{i \in \omega} D_{B_i}$, hence $\{b \mid \langle a, b \rangle \in \bigcup_{i \in \omega} B_i\}$ is at most countable. Thus $\bigcup_{i \in \omega} B_i \neq X$.

Questions about Borel partitions of subsets of $\mathbb{R}$ of power $\aleph_1$, are equivalent to questions about general partitions of $\aleph_1$. Galvin and Shelah deal with such questions in [6]. This fact is expressed in the following observation, which is due independently to K. Kunen, B.V. Rao, and J. Silver.

Observation 1.9. *Let R be an n-place relation on $\aleph_1$. Then there is a G_δ relation S on the Cantor set C and a subset A of C such that $\langle A, S \restriction A\rangle \cong \langle \aleph_1, R\rangle$.*

Proof. For the sake of simplicity we take an R which is binary symmetric and irreflexive. We represent C as ${}^{\omega}5$. Let $\{a^\alpha \mid \alpha < \aleph_1\}$ be a family of almost disjoint infinite subsets of ω. For every $\alpha < \aleph_1$, let $\{a^\alpha_\beta \mid \beta \leqslant \alpha\}$ be a family of pairwise disjoint subsets of ω such that for every $\beta \leqslant \alpha$ the symmetric difference of a^α_β and a^β is finite. Let $\{b^\alpha \mid \alpha < \aleph_1\}$ be a family of infinite subsets of ω such that for every $\alpha < \beta < \aleph_1$, $b^\alpha - b^\beta$ is finite and $b^\beta - b^\alpha$ is infinite. For every $\alpha < \aleph_1$ we define $\eta_\alpha \in {}^{\omega}5$. $\eta_\alpha(2i+1) = 1$ if $i \in b^\alpha$ and otherwise $\eta_\alpha(2i+1) = 0$. $\eta_\alpha(2i) = 2$ if $i \in a^\alpha_\alpha$, $\eta_\alpha(2i) = 3$ if for some $\beta < \alpha$, $i \in a^\alpha_\beta$ and $\langle \beta, \alpha\rangle \in R$. Otherwise $\eta_\alpha(2i) = 4$. Let $S_1 = \{\langle \eta, \nu\rangle \mid \eta, \nu \in {}^{\omega}5, \{i \mid \nu(i) = 1 \text{ and } \eta(i) = 0\}$ is infinite and $\{i \mid \nu(i) = 3$ and $\eta(i) = 2\}$ is infinite$\}$, and let $S = S_1 \cup S_1^{-1}$. Let $A = \{\eta_\alpha \mid \alpha < \aleph_1\}$; clearly S is a G_δ set and $\langle A, S \restriction A\rangle \cong \langle \aleph_1, R\rangle$. □

Question 1.10. Using oracle forcing it is easy to construct a model of set theory in which $\mathbb{R}$ contains a second category set of power $\aleph_1$, and in which for every second countable space Y of the second category and every SOC of Y, Y contains an uncountable f-homogeneous subset. We do not know whether

$$\text{SOCA} + (\exists X \subseteq \mathbb{R})\ (|X| = \aleph_1 \text{ and } X \text{ is of the second category})$$

is consistent.

Question 1.11. If in SOCA one replaces everywhere $\aleph_1$ by $\aleph_2$ is the resulting axiom still consistent?

Question 1.12. If in observation 1.9 one replaces $\aleph_1$ by $\aleph_2$ is the resulting statement consistent with ZFC?

2. The explicit contradiction method, and the increasing set axiom

Suppose that we want to construct a model of $\text{MA} + \aleph_1 < 2^{\aleph_0}$ or of $\text{MA}_{\aleph_1}$, and at the same time we want to preserve a certain property Φ of a certain set A. There is a problem when we encounter a c.c.c. forcing set P which ruins property Φ, that is, in V^P, A does not satisfy Φ anymore. In such a case we shall find a c.c.c. forcing set Q such that in V^Q, P is not c.c.c., and A still has property Φ. We call the particular method in which we do this 'the explicit contradiction method'.

We take the liberty to explain this method by an application which yields a known result. We do so in order not to start with applications that involve more than one tehcnique.

Definition. Let $A \subseteq \mathbb{R}$ be of power $\aleph_1$; A is called an *increasing set*, if in every uncountable set of pairwise disjoint finite sequences from A there are two sequences $\langle a_1, \ldots, a_n\rangle$, $\langle b_1, \ldots, b_n\rangle$ having the same length such that $a_1 < b_1, \ldots, a_n < b_n$.

Axiom ISA. These exists an increasing set.

The following theorem is due to Avraham and Shelah [1]. It follows from Theorem 2 there, together with the discussion preceding it.

Theorem 2.1. $\mathrm{MA}_{\aleph_1} + \mathrm{ISA}$ *is consistent.*

Remark. The proof in [1] is slightly different from ours and does not use the explicit contradiction method. Instead, there, a model V^P is constructed such that $V^P \models$"Every uncountable 1-1 function from A to A contains an uncountable OP subfunction". This implies that there is no c.c.c. $Q \in V^P$ such that in V^P, $\Vdash_Q A$ is not increasing.

This slight difference between the proof becomes essential, if one wants at the same time to carry out some task that requires CH in the intermediate stages.

E.g., **Theorem.** MA *is consistent with the existence of a rigid increasing set.* ('Rigid' means there are no order automorphisms except for the identity.)

Proof. Let V be any universe. Let us add to V a set A of $\aleph_1$ Cohen reals. It is easy to see that in this Cohen extension of V the set A is increasing. (This fact and more appears in [1, §5 Remark 21].) Hence we can w.l.o.g. assume that this is our universe V and there is an increasing set A.

As usual we will define a finite support iteration $\{P_i \mid i < 2^{\aleph_1}\}$ in which all possible c.c.c. forcing sets of power $\aleph_1$ are considered. For each single step in the iteration we need the following lemma in which the explicit contradiction method is used.

Lemma 2.2. *Let A be an increasing set in V, P be a c.c.c. forcing set $p \in P$, and $p \Vdash_P$ "A is not increasing". Then there is a c.c.c. forcing set $Q = Q_{P,p,A}$ of power $\aleph_1$ such that $\Vdash_Q$ "A is increasing and P is not c.c.c.".*

Proof. Let $\tilde{B}$ be a P-name of a set of pairwise disjoint 1-1 sequences of length n such that $p \Vdash_P$ "$\tilde{B}$ is a counterexample to the increasingness of A". Since P is c.c.c. it is easy to pick a sequence $\{\langle p_i, \boldsymbol{b}^i\rangle \mid i < \aleph_1\}$ such that: (1) for $i : p_i \geq p$ and $p_i \Vdash_P \boldsymbol{b}^i \in \tilde{B}$ and (2) let $\boldsymbol{b}^i = \langle b^i_1, \ldots, b^i_n\rangle$, then for every $i \neq j$, $\{b^i_1, \ldots, b^i_n\} \cap \{b^j_1, \ldots, b^j_n\} = \emptyset$.

Let $\boldsymbol{a} = \langle a_1, \ldots, a_n\rangle$, $\boldsymbol{b} = \langle b_1, \ldots, b_n\rangle \in \mathbb{R}^n$ be distinct, we say that $\{\boldsymbol{a}, \boldsymbol{b}\}$ is order preserving (OP), if for every $1 \leq k \leq l \leq n$ $a_k < b_k \Leftrightarrow a_l < b_l$. We say that p_i, p_j are explicitly contradictory if $\{\boldsymbol{b}^i, \boldsymbol{b}^j\}$ is OP. The main point is that if p_i and p_j are explicitly contradictory, we indeed know that they are incompatible in P; for if $r \geq p_i, p_j$, then $r \geq p$, hence $r \Vdash_P$ "$\tilde{B}$ is a counterexample to the increasingness of A, and $\boldsymbol{b}^i, \boldsymbol{b}^j \in B$". This is of course a contradiction. Recall that we are looking for a Q that will add an uncountable anti-chain to P. Hence our choice for Q is obvious. Let $\sigma \in P_{\aleph_0}(\aleph_1)$ and $q_\sigma \stackrel{\text{def}}{=} \{p_i \mid i \in \sigma\}$. Let $Q' = \{q_\sigma \mid \sigma \in p_{\aleph_0}(\aleph_1)$ and for every $i \neq j \in \sigma$, p_i and p_j are explicity contradictory$\}$. $q_\tau \leq q_\sigma$ if $\tau \subseteq \sigma$.

Obviously a Q'-generic set adds an antichain D to P. Once we show that Q' is c.c.c., there is a standard way to find some $q_0 \in Q'$ such that $q_0 \Vdash_{Q'}$ "D is uncountable". Hence we shall take Q to be $\{q \in Q' \mid q_0 \leq q\}$.

We thus show that Q' is c.c.c. Let $\{q_{\sigma_i} \mid i < \aleph_1\}$ be an uncountable subset of Q'. W.l.o.g. $\{\sigma_i \mid i < \aleph_1\}$ is a Δ-system and for every i, $\sigma_i = \{\alpha^1, \ldots, \alpha^k, \alpha^{i,1}, \ldots, \alpha^{i,l}\}$ where $\alpha^1 < \cdots < \alpha^k < \alpha^{i,1} < \cdots < \alpha^{i,l}$. Let $\boldsymbol{c} = \boldsymbol{b}^{\alpha^1} \frown \cdots \frown \boldsymbol{b}^{\alpha^k} \stackrel{\text{def}}{=} \langle c_1, \ldots, c_m \rangle$, and $\boldsymbol{c}^i = \boldsymbol{b}^{\alpha^{i,1}} \frown \cdots \frown \boldsymbol{b}^{\alpha^{i,l}} \stackrel{\text{def}}{=} \langle c_1^i, \ldots, c_r^i \rangle$. For every β let $U_1^\beta, \ldots, U_r^\beta$ be rational neighborhoods of $c_1^\beta, \ldots, c_r^\beta$ respectively such that for every $1 \leq i$, $j \leq r$ for every $d_i \in U_i^\beta$ and $d_j \in U_j^\beta$: $c_i^\beta < c_j^\beta \Leftrightarrow d_i < d_j$. By choosing a subsequence, we can assume that for every i, U_i^β is independent of β.

Let β, γ be such that $\{\boldsymbol{c}^\beta, \boldsymbol{c}^\gamma\}$ is OP. Hence for every $1 \leq i \leq k$, $\{\boldsymbol{b}^{\alpha^{\beta,i}}, \boldsymbol{b}^{\alpha^{\gamma,i}}\}$ is OP. If $i \neq j$, then since $\{\boldsymbol{b}^{\alpha^{\beta,i}}, \boldsymbol{b}^{\alpha^{\beta,j}}\}$ is OP, and since we uniformized the U_i^β's, also $\{\boldsymbol{b}^{\alpha^{\beta,i}}, \boldsymbol{b}^{\alpha^{\gamma,j}}\}$ is OP. Hence $\{q_{\sigma_\beta} \cup q_{\sigma_\gamma}\} \in Q'$. So Q' is c.c.c.

Our next goal is to show that $\Vdash_{Q'}$ "A is increasing". The proof is very similar to the proof that Q' is c.c.c. Suppose by contradiction $\tilde{B}$ is a Q'-name, $q_0 \in Q'$ and $q_0 \Vdash_{Q'}$ "$\tilde{B}$ is a name of a counterexample to the increasingness of A". Let $\{\langle q_{\sigma_\alpha}, \boldsymbol{a}^\alpha \rangle \mid \alpha < \aleph_1\}$ be a sequence such that for every α, $q_0 \leq q_{\sigma_\alpha}$, $q_{\sigma_\alpha} \Vdash \boldsymbol{a}^\alpha \in \tilde{B}$, and for every $\alpha \neq \beta$, $\boldsymbol{a}^\alpha$ and $\boldsymbol{a}^\beta$ are disjoint. As in the previous argument we assume that the σ_α's form a Δ-system, and we choose $U_i^\alpha = U_i$ with the same properties. Define the $\boldsymbol{c}^\alpha$'s as in the previous argument, and find β, γ such that $\{\boldsymbol{c}^\beta \frown \boldsymbol{d}^\beta, \boldsymbol{c}^\gamma \frown \boldsymbol{d}^\gamma\}$ is OP; then $q_{\sigma_\beta} \cup q_{\sigma_\gamma} \in Q'$, $\{\boldsymbol{d}^\beta, \boldsymbol{d}^\gamma\}$ is OP, a contradiction. □

Continuation of the proof of Theorem 2.1. It follows from Lemma 2.2 that if P is a c.c.c. forcing set such that $\Vdash_P$ "A is increasing", and if $\tilde{Q}$ is a P-name of a c.c.c. forcing set, then there is a P-name $\tilde{R} = \tilde{R}_{\tilde{Q}}$ such that $\Vdash_P$ ($\tilde{R}$ is a c.c.c. forcing set and $\Vdash_{\tilde{R}}$ "A is increasing"), and for every $p \in P$: if $p \Vdash_P$ ($\Vdash_{\tilde{Q}} A$ is increasing), then $p \Vdash_P \tilde{R} = \tilde{Q}$; and if $p \Vdash_P (\exists q \in \tilde{Q})$ ($q \Vdash_Q A$ is not increasing), then $p \Vdash_P (\Vdash_{\tilde{R}} \tilde{Q}$ is not c.c.c.).

Let $\{N_i \mid i < 2^{\aleph_1}\}$ be an enumeration of $P_{\aleph_2}(2^{\aleph_1})$. We define by induction an increasing sequence of forcing sets. P_0 is the trivial forcing set, and if δ is a limit ordinal, then $P_\delta = \bigcup_{i<\delta} P_i$. Suppose P_i has been defined; if $\nVdash_{P_i}$ "A is increasing" or if $\nVdash_{P_i}$ "N_i is a c.c.c. forcing set", then let $P_{i+1} = P_i$; otherwise let $P_{i+1} = P_i * \tilde{R}_{N_i}$.

We first show that for every i, $\Vdash_{P_i}$ "A is increasing". By our definition if this happens for P_i, then it happens for P_{i+1}.

Let $\mathrm{cf}(\delta) > \aleph_0$, and suppose for every $i < \delta$, $\Vdash_{P_i}$ "A is increasing". Suppose by contradiction that G is a P_δ-generic set and $B \in V[G]$ is a counterexample to the increasingness of A. Let $G_i = G \cap P_i$. There is $i < \delta$ such that the closure of B in A^n, $\bar{B}$, belongs to $V[G_i]$. It is easy to see that there is $B' \in V[G_i]$ such that B' is an uncountable subset of $\bar{B}$ consisting of pairwise disjoint sequences. Hence there cannot be two sequences in B' which form an OP pair. Hence A is not increasing in $V[G_i]$, a contradiction.

Let $\mathrm{cf}(\delta) = \aleph_0$, suppose our claim is true for every $i < \delta$. Let G be a P_δ-generic set and let $B \subseteq A^n$ and $B \in V[G]$. Then there are $\{B_i \mid i \in \omega\}$ such that $\bigcup_{i\in\omega} B_i = B$ and for every i there is $\gamma_i < \delta$ such that $B_i \in V[G_{\gamma_i}]$. Hence one of the B_i's is uncountable, hence if B is a counterexample to the increasingness of A, there is such an example belonging to a previous $V[G_i]$, and by the induction hypothesis this is impossible.

Let $P = P_{2^{\aleph_1}}$, the argument showing that $\mathrm{MA}_{\aleph_1}$ holds in V^P is standard. □

Remark. Note that if $\mathrm{MA}_{\aleph_1}$ holds, then A is increasing iff for every 1-1 uncountable $f \subseteq A \times A$ there is an uncountable OP function $g \subseteq f$.

3. The open coloring axiom, and how to preassign colors

In [1] it was shown (Theorem 6) that it is consistent with ZFC that every 1-1 $f \subseteq \mathbb{R} \times \mathbb{R}$ of power $\aleph_1$ is the union of countably many monotonic functions. This fact is a special case of the open coloring axiom (OCA) to be defined below. (S. Todorčević proved that, under MA, OCA is a consequence of this fact.)

Let X be a second countable Hausdorff space of power $\aleph_1$. An *open coloring* of X is finite cover $\mathcal{U} = \{U_0, \dots, U_{n-1}\}$ of $D(X)$ such that for every l, $U_l = \{\langle y, x\rangle \mid \langle x, y\rangle \in U_l\}$. $A \subseteq X$ is $\mathcal{U}$-homogeneous if for some color l, $D(A) \subseteq U_l$. A $\mathcal{U}$-homogeneous partition of X is a countable partition $\{X_i \mid i \in \omega\}$ of X consisting of $\mathcal{U}$-homogeneous sets.

The open coloring axiom is as follows.

Axiom OCA. For every X and every open coloring $\mathcal{U}$ of X, X has a $\mathcal{U}$-homogeneous partition.

It turns out that in a universe V satisfying MA+OCA+ISA, the set of real order types of power $\aleph_1$ has nice properties, e.g. there are exactly three homogeneous such order types; so our first goal is to prove the consistency of the conjunction of these three axioms. As seen in the following theorem we do a little more, and add to the above axioms also SOCA.

Theorem 3.1. MA+OCA+SOCA+ISA *is consistent.*

Later in the section we shall prove a generalization of OCA.

Proof of Theorem 3.1. We start with a universe V satisfying CH+$(2^{\aleph_1}=\aleph_2)$ and with an increasing set $A\in V$. We construct a finite support iteration $\{P_i \mid i\leqslant\aleph_2\}$, according to a list of tasks of length $\aleph_2$ which is prepared in advance. In each atomic step of the iteration we deal with one of the following tasks.

(1) For a given c.c.c. forcing set Q of power $\aleph_1$, we have to find a c.c.c. forcing set $P=P_Q$ of power $\aleph_1$ such that $\Vdash_P$ "A is increasing", and either Q is not c.c.c. or there is a Q-generic filter over V.

(2) For a given X and a SOC f of X we have to find a c.c.c. forcing set $P=P_{X,f}$ of power $\aleph_1$ such that $\Vdash_P$ "A is increasing", and X contains an uncountable f-homogeneous subset.

(3) For a given X and an open coloring $\mathcal{U}$ of X we have to find a c.c.c. forcing set $P=P_{X,\mathcal{U}}$ of power $\aleph_1$ such that $\Vdash_P$ "A is increasing", and X has a $\mathcal{U}$-homogeneous partition.

We expect the reader to known how to define the list of tasks, how to define the iteration and why $V[P_{\aleph_2}]$ satisfies all the four axioms. We shall concentrate only on the atomic steps of the iteration.

The existence of $P=P_Q$ satisfying the requirements of (1) was proved in the previous section (Lemma 2.2).

We start with task (3) where the additional trick of preassigning colors is used. This method appears also in [1, Theorem 6]. There, a special case of OCA is proved. In the present application there is an additional complication, since at the same time we want to preserve the increasingness of A.

Lemma 3.2. *Suppose $V\vDash$"CH, $A\in V$ is increasing" and $\mathcal{U}=\{U_0,\ldots,U_{n-1}\}$ is an open coloring of X. Then there is a c.c.c. forcing set P of power $\aleph_1$ such that $\Vdash_P$ "A is increasing", and X has a $\mathcal{U}$-homogeneous partition.*

Proof. W.l.o.g. $A, X\subseteq\aleph_1$. As in Theorem 1.1 we form a model M with universe $\aleph_1$ that includes enough set theory, and includes also A, X and $\mathcal{U}$ as predicates. Let M_α be the submodel of M whose universe is α, and let $C=C_M=\{\alpha \mid M\cap\alpha\prec M\}$.

We know that each element of X should be put into one of a countable set of homogeneous subsets of X, and our first aim is to decide in advance what will be the color of the homogeneous set to which each element a of X should belong. Let $\{\alpha_i \mid i<\aleph_1\}$ be an isomorphism between $\langle\aleph_1,<\rangle$ and $\langle C,<\rangle$, let $E_i=\{\beta \mid \alpha_i\leqslant\beta<\alpha_{i+1}\}$, and let $\mathscr{E}=\{E_i \mid i<\aleph_1\}$; we call $\mathscr{E}$ the set of C-slices. For every $i<\aleph_1$ let $\{a^i_l \mid l\in\omega\}$ be an enumeration of $X\cap E_i$ such that $a^i_0=\min(X\cap E_i)$. Let $\varphi(x)=\varphi(x_1,\ldots,x_l)$ be a formula in the language of M and possibly with parameters from $|M|$, $Q\mathbf{x}\,\varphi(\mathbf{x})$ abbreviates the following formula $\forall\alpha\,(\exists x_1>\alpha)\cdots(\exists x_l>\alpha)\,\varphi(\mathbf{x})$. Let $\boldsymbol{\delta}=\langle\delta_0,\ldots,\delta_{l-1}\rangle\in{}^l n$ be a sequence, and $\varphi(\mathbf{x},\mathbf{y})=$

$\varphi(x_0, \dots, x_{l-1}, y_1, \dots, y_m)$ be a formula with parameter from $|M|$; we denote

$$\psi_{\varphi,\boldsymbol{\delta}} \equiv \varphi(\boldsymbol{x}, \boldsymbol{y}) \wedge \varphi(\boldsymbol{x}', \boldsymbol{y}') \wedge \left(\bigwedge_{t=0}^{l-1} x_t, x'_t \in X\right) \wedge \left(\bigwedge_{t=1}^{m} y_t, y'_t \in A\right)$$
$$\wedge \left(\bigwedge_{t=0}^{l-1} \langle x_t, x'_t\rangle \in U_{\delta_t}\right) \wedge (\{\boldsymbol{y}, \boldsymbol{y}'\} \text{ is OP})$$

where $\boldsymbol{x}'$, $\boldsymbol{y}'$ are disjoint sequences of distinct variables disjoint from $\boldsymbol{x}$ and $\boldsymbol{y}$.

Claim 1. *Let $i < \aleph_1$. Then for every $l \in \omega$ there is $\boldsymbol{\delta} \in {}^l n$ such that for every $m \in \omega$ and every $\varphi(\boldsymbol{x}, \boldsymbol{y}) \equiv \varphi(x_0, \dots, x_{l-1}, y_1, \dots, y_m)$ with parameters from $|M_{\alpha_i}|$: if there are $b_1, \dots, b_m \in A \cap (|M| - |M_{\alpha_i}|)$ such that $M \vDash \varphi[a^i_0, \dots, a^i_{l-1}, b_1, \dots, b_m]$, then $M \vDash Q\boldsymbol{x}, \boldsymbol{y}\, Q\boldsymbol{x}', \boldsymbol{y}'\, \psi_{\varphi,\boldsymbol{\delta}}$.*

Proof. Suppose by contradiction the claim is not true, so for every $\boldsymbol{\delta} \in {}^l n$ let $\varphi_{\boldsymbol{\delta}}(\boldsymbol{x}, \boldsymbol{y}^{\boldsymbol{\delta}})$ be a formula showing that $\boldsymbol{\delta}$ is not as required in the claim. We assume that the $\boldsymbol{y}^{\boldsymbol{\delta}}$'s are pairwise disjoint sequences of variables, and that their concatenation is $\boldsymbol{y} = \langle y_1, \dots, y_m\rangle$. Let

$$\varphi(\boldsymbol{x}, \boldsymbol{y}) \equiv \bigwedge_{\boldsymbol{\delta} \in {}^l n} \varphi_{\boldsymbol{\delta}}, \quad \text{and} \quad \chi(\boldsymbol{x}, \boldsymbol{y}) \equiv \varphi(\boldsymbol{x}, \boldsymbol{y}) \wedge \left(\bigwedge_{t=1}^{m} y_t \in A\right).$$

By the choice of the $\varphi_{\boldsymbol{\delta}}$'s there are $b_1, \dots, b_m \in A \cap (|M| - |M_{\alpha_i}|)$ such that $M \vDash \varphi[a^i_0, \dots, a^i_{l-1}, b_1, \dots, b_m]$, hence

(1) $M \vDash Q\boldsymbol{x}, \boldsymbol{y}\, \chi(\boldsymbol{x}, \boldsymbol{y})$.

On the other hand it is clear that for every $\boldsymbol{\delta}$

(2) $M \vDash \neg Q\boldsymbol{x}, \boldsymbol{y}\, Q\boldsymbol{x}', \boldsymbol{y}'\, \psi_{\varphi,\boldsymbol{\delta}}$.

Hence there is $\beta^0 < \aleph_1$ such that for every $\boldsymbol{a} \in |M|^l$ and $\boldsymbol{b} \in |M|^m$ if $\beta^0 < \boldsymbol{a}, \boldsymbol{b}$ then there is $\beta = \beta(\boldsymbol{a}, \boldsymbol{b})$ such that for every $\boldsymbol{\delta} \in {}^l n$ and $\boldsymbol{a}', \boldsymbol{b}' > \beta$, $M \vDash \neg\psi_{\varphi,\boldsymbol{\delta}}[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a}', \boldsymbol{b}']$.

We define by induction on $j < \aleph_1$, $\boldsymbol{a}^j \in X^l$ and $\boldsymbol{b}^j \in A^m$, our induction hypothesis is that for every $j < \aleph_1$, $\boldsymbol{a}^j, \boldsymbol{b}^j > \beta_0$. Suppose $\boldsymbol{a}^k$, $\boldsymbol{b}^k$ have been defined for every $k < j$. Let $\beta_j > \beta^0 \cup \bigcup_{k<j} \beta(\boldsymbol{a}^k, \boldsymbol{b}^k)$. Let $\boldsymbol{a}^j \in X^l$, $\boldsymbol{b}^j \in A^m$ be such that $\boldsymbol{a}^j, \boldsymbol{b}^j > \beta_j$, and $M \vDash \varphi[\boldsymbol{a}^j, \boldsymbol{b}^j]$. This choice is possible by (1).

By the increasingness of A there are $k < j$ such that $\{\boldsymbol{b}^k, \boldsymbol{b}^j\}$ is OP. Let $\boldsymbol{a}^j = \langle a^{j,0}, \dots, a^{j,l-1}\rangle$ and $\boldsymbol{a}^k = \langle a^{k,0}, \dots, a^{k,l-1}\rangle$, and let δ_t be such that $\langle a^{j,t}, a^{k,t}\rangle \in U_{\delta_t}$. Let $\boldsymbol{\delta} = \langle \delta_0, \dots, \delta_{l-1}\rangle$. $M \vDash \chi[\boldsymbol{a}^k, \boldsymbol{b}^k]$, and $M \vDash \chi[\boldsymbol{a}^j, \boldsymbol{b}^j]$; however since $\boldsymbol{a}^j, \boldsymbol{b}^j > \beta_j$, $M \vDash \neg\psi_{\varphi,\boldsymbol{\delta}}[\boldsymbol{a}^k, \boldsymbol{b}^k, \boldsymbol{a}^j, \boldsymbol{b}^j]$. This is a contradiction, and the claim is proved. □

Let $i < \aleph_1$; for every $l \in \omega$ let $\boldsymbol{\delta}^i_l$ be the least element in ${}^l n$ according to the lexicographic order of ${}^l n$, which satisfies the requirements of Claim 1. Recalling that for every $l \in \omega$, $a^i_0 \leq a^i$, it is easy to see that if $k < l$, then $\boldsymbol{\delta}^i_k$ is an initial segment of $\boldsymbol{\delta}^i_l$. Let $\langle \delta^i_0, \delta^i_1, \dots\rangle = \bigcup_{l\in\omega} \boldsymbol{\delta}^i_l$. If $a \in X$ and $a \geq \alpha_0$, then for some i and l, $a = a^i_l$; we denote $\delta(a) = \delta^i_l$ and call $\delta(a)$ the color of a. We have thus assigned a color to every a in $X-\alpha_0 \overset{\text{def}}{=} X'$, and when we construct P we shall put each $a \in X'$ in a homogeneous set of color $\delta(a)$.

We are ready to define the forcing set P which satisfies the requirements of the lemma.

Let $\{n_i \mid i \in \omega\}$ be an enumeration of the set of colors n such that for every $l<n$, $\{i \mid n_i = l\}$ is infinite. Let P be the set of finite approximations of a homogeneous partition $\{X_i \mid i \in \omega\}$ of X' which respects the preassigned colors, and in which X_i has the color n_i. More precisely, $P=\{f \mid \text{Dom}(f) \in P_{\aleph_0}(X'),\ \text{Rng}(f) \subseteq \omega$, and for every $a, b \in \text{Dom}(f)$ if $f(a)=f(b)=i$, then $\langle a, b\rangle \in U_{n_i}$ and $\delta(a)=\delta(b)=n_i\}$.

Clearly $\Vdash_P$ "X has a U-homogeneous partition". We have to show that P is c.c.c., and that $\Vdash_P$ "A is increasing". The proofs of these two facts are similar, we thus skip the first, and assuming that we already know that P is c.c.c., we prove that $\Vdash_P$ "A is increasing".

Suppose by contradiction there is $p^0 \in P$ and $m \in \omega$ such that $p^0 \Vdash_P$ "There is a family $\{\boldsymbol{b}^\alpha \mid \alpha<\aleph_1\} \subseteq A^m$ of pairwise disjoint sequences such that for no $\alpha \neq \beta$, $\{\boldsymbol{b}^\alpha, \boldsymbol{b}^\beta\}$ is OP". Let $\tilde{B}$ be a name for this family. Let $\{\langle p_\alpha, \boldsymbol{b}^\alpha\rangle \mid \alpha<\aleph_1\}$ be such that (1) $p_\alpha \geqslant p^0$; (2) $p_\alpha \Vdash_P \boldsymbol{b}^\alpha \in \tilde{B}$; and (3) if $\alpha \neq \beta$, then $\boldsymbol{b}^\alpha$ and $\boldsymbol{b}^\beta$ are pairwise disjoint.

W.l.o.g. the p_α's form a Δ-system, and they all have the same structure. More precisely, we need the following uniform behavior of the $\langle p_\alpha, \boldsymbol{b}^\alpha\rangle$'s.

(1) $\{\text{Dom}(p_\alpha) \mid \alpha<\aleph_1\}$ is a Δ-system.

(2) Let $\text{Dom}(p_\alpha)=\{a_{\alpha,0}, \ldots, a_{\alpha,l}\}$ where $a_{\alpha,0}<\cdots<A_{\alpha,l}$ and the first r elements form the kernel of $\{\text{Dom}(p_\alpha) \mid \alpha<\aleph_1\}$ and $\boldsymbol{b}^\alpha=\langle b_{\alpha,1}, \ldots, b_{\alpha,m}\rangle$. Then for every $\alpha, \beta<\aleph_1$ for every $i, j \leqslant l$ and $1 \leqslant k$, $t \leqslant m: \delta(a_{\alpha,i})=\delta(a_{\beta,i})$, $p_\alpha(a_{\alpha,i})=p_\beta(a_{\beta,i})$, $b_{\alpha,k}=a_{\alpha,i}$ iff $b_{\beta,k}=a_{\beta,i}$ and $b_{\alpha,k} \leqslant b_{\alpha,t}$ iff $b_{\beta,k} \leqslant b_{\beta,t}$.

We can assume that for every $\alpha<\aleph_1$, $b_{\alpha,1}<\cdots<b_{\alpha,m}$.

Since X is second countable, we can further uniformize the $\langle p_\alpha, \boldsymbol{b}^\alpha\rangle$'s in the following way.

(3) There are open sets $V_0, \ldots, V_l \subseteq X$ such that for every $\alpha<\aleph_1$ and distinct $i, j \in \{0, \ldots, l\}: a_{\alpha,i} \in V_i$, if $p_\alpha(a_{\alpha,i})=p_\alpha(a_{\alpha,j})$ then $V_i \times V_j \subseteq U_{\delta(a_{\alpha,i})}$.

Let $\boldsymbol{d}_\alpha=\langle a_{\alpha,0}, \ldots, a_{\alpha,l}, b_{\alpha,1}, \ldots, b_{\alpha,m}\rangle$, and let D be the topological closure of $\{\boldsymbol{d}_\alpha \mid \alpha<\aleph_1\}$ in $X^{l+1} \times A^m$. Since X is second countable D is the closure of a countable set, hence it is definable by a parameter d in M. Let $\gamma \in C_M$ be such that $d, a_{\alpha,0}, \ldots, a_{\alpha,r-1} \in |M_\gamma|$, (recall that $\{a_{\alpha,0}, \ldots, a_{\alpha,r-1}\}$ is the kernel of $\{\text{Dom}(p_\beta) \mid \beta<\aleph_1\}$.) We choose α such that $a_{\alpha,r}, \ldots, a_{\alpha,l}, b_{\alpha,1}, \ldots, b_{\alpha,l} \notin |M_\gamma|$. We intend to apply the duplication argument to $\boldsymbol{d}_\alpha$.

Let $\boldsymbol{d}_0=\langle a_0, \ldots, a_l, b_1, \ldots, b_m\rangle$, $\boldsymbol{a}=\langle a_0, \ldots, a_l\rangle$ and $\boldsymbol{b}=\langle b_1, \ldots, b_m\rangle$. Let $E^1, \ldots, E^k$ be those C_M-slices E for which there is a_i, $r \leqslant i \leqslant l$, such that $a_i \in E$, or there is b_j, $1 \leqslant j \leqslant m$, such that $b_j \in E$. Let $\boldsymbol{a}=\boldsymbol{a}^0 \frown \cdots \frown \boldsymbol{a}^k$ where $\boldsymbol{a}^0=\langle a_0, \ldots, a_{r-1}\rangle$ is the sequence of those elements of $\boldsymbol{a}$ which belongs to $|M_\gamma|$, and for $i>0$, $\boldsymbol{a}^i$ is the sequence of those elements of $\boldsymbol{a}$ which belong to E^i. Let $\boldsymbol{b}=\boldsymbol{b}^1 \frown \cdots \frown \boldsymbol{b}^k$ where $\boldsymbol{b}^i$ is the sequqnce of those elements of $\boldsymbol{b}$ which belong to E^i. Let β_i be the minimal element of E^i.

We define by a downward induction formulas $\varphi_k, \ldots, \varphi_0$. The induction hypotheses are: (1) the parameters of φ_i belong to $|M_\gamma|$, and (2) $M \models \varphi_i[\boldsymbol{a}^1, \ldots, \boldsymbol{a}^i, \boldsymbol{b}^1, \ldots, \boldsymbol{b}^i]$. Let $\varphi_0 \equiv \langle a_0, \ldots, a_{r-1}\rangle^\frown \cdots ^\frown \boldsymbol{x}^k {}^\frown \boldsymbol{y}^1 {}^\frown \cdots ^\frown \boldsymbol{y}^k \in D$. Suppose that $\varphi_{i+1}(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^{i+1}, \boldsymbol{y}^1, \ldots, \boldsymbol{y}^{i+1})$ has been defined. Let s be the length of $\boldsymbol{b}^{i+1}$, let $\boldsymbol{a}^{i+1} = \langle a'_1, \ldots, a'_t\rangle$, and let $\delta_j = \delta(a'_j)$, $j = 1, \ldots, t$. The formula $\varphi_{i+1}(\boldsymbol{a}^1, \ldots, \boldsymbol{a}^i, \boldsymbol{x}^{i+1}, \boldsymbol{b}^1, \ldots, \boldsymbol{b}^i, \boldsymbol{y}^{i+1})$ has parameters from $|M_{\beta_{i+1}}|$, hence by the definition of δ there are $c^l = \langle c^l_1, \ldots, c^l_t\rangle$ and $\boldsymbol{d}^l$, $l = 1, 2$, such that: (1) $M \models \varphi_{i+1}[\boldsymbol{a}^1, \ldots, \boldsymbol{a}^i, \boldsymbol{c}^l, \boldsymbol{b}^1, \ldots, \boldsymbol{b}^i, \boldsymbol{d}^l]$, $l = 1, 2$; (2) for every $j = 1, \ldots, t$, $\langle c^1_j, c^2_j\rangle \in U_{\delta_j}$; and (3) $\{\boldsymbol{d}^1, \boldsymbol{d}^2\}$ is OP.

For $l = 1, 2$, $j = 1, \ldots, t$ let $V^{i+1,l}_j$ be basic open sets in X such that $\langle c^1_j, c^2_j\rangle \in V^{i+1,1}_j \times V^{i+1,2}_j \subseteq U_{\delta_j}$; and let $V^{i+1,l} = V^{i+1,l}_1 \times \cdots \times V^{i+1,l}_t$. For $l = 1, 2$ let $W^{i+1,l}$ be basic open sets in A^s such that $\boldsymbol{d}^l \in W^{i+1,l}$, $l = 1, 2$, and for every $\boldsymbol{d}_1 \in W^{i+1,1}$ and $\boldsymbol{d}_2 \in W^{i+1,2}$, $\{\boldsymbol{d}_1, \boldsymbol{d}_2\}$ is OP. Let $\boldsymbol{u}^l = \langle u^l_1, \ldots, u^l_t\rangle$, $\boldsymbol{v}^l = \langle v^l_1, \ldots, v^l_s\rangle$ be sequences of variables. Let

$$\begin{aligned}&\varphi_i(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^i, \boldsymbol{y}^1, \ldots, \boldsymbol{y}^i)\\ &\equiv \exists \boldsymbol{u}^1, \boldsymbol{u}^2, \boldsymbol{v}^1, \boldsymbol{v}^2 \left(\bigwedge_{l=1}^{2} \varphi_{i+1}(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^i, \boldsymbol{u}^l, \boldsymbol{y}^1, \ldots, \boldsymbol{y}^i, \boldsymbol{v}^l)\right)\\ &\wedge \left(\bigwedge_{l=1}^{2} \boldsymbol{u}^l \in V^{i+1,l}\right) \wedge \left(\bigwedge_{l=1}^{2} \boldsymbol{v}^l \in W^{i+1,l}\right).\end{aligned}$$

Clearly φ_i satisfies the induction hypotheses. We have thus defined φ_0.

As it was done in Theorem 1.1 starting with φ_0 we can choose two sequences $\boldsymbol{a}^{1,l} {}^\frown \cdots ^\frown \boldsymbol{a}^{k,l} {}^\frown \boldsymbol{b}^{1,l} {}^\frown \cdots ^\frown \boldsymbol{b}^{k,l} \stackrel{\text{def}}{=} \boldsymbol{e}^l$, $l = 1, 2$, such that $\boldsymbol{a}^0 {}^\frown e^1, \boldsymbol{a}^0 {}^\frown e^2 \in D$, for every $i = 1, \ldots, k$, $\boldsymbol{a}^{i,1} {}^\frown \boldsymbol{a}^{i,2} \in V^{i,1} \times V^{i,2}$ and $\boldsymbol{b}^{i,1} {}^\frown \boldsymbol{b}^{i,2} \in W^{i,1} \times W^{i,2}$.$D_0 = \{\boldsymbol{d}_\alpha \in \alpha < \aleph_1\}$ was dense in D, hence there are $d_{\alpha(l)} \in D_0$, $l = 1, 2$, such that

$$d_{\alpha(l)} \in X^{r-1} \times V^{1,l} \times \cdots V^{k,l} \times W^{1,l} \times \cdots W^{k,l}.$$

It is easy to see that $p_{\alpha(1)} \cup p_{\alpha(2)} \in P$ and $\{\boldsymbol{b}^{\alpha(1)}, \boldsymbol{b}^{\alpha(2)}\}$ is OP. This contradicts the assumption that $p^0 \Vdash_P$ "$\tilde{B}$ is a counter-example to the increasingness of A". Hence Lemma 3.2 is proved. □

We turn now to the last kind of tasks that we have got to carry out.

Lemma 3.3 (CH). *Let $A \in V$ be an increasing set, and let f be a SOC of X, then there is a c.c.c. forcing set P of power $\aleph_1$ such that $\Vdash_P$ "A is increasing, and X contains an f-homogeneous uncountable subset".*

Proof. Let us first assume that (*): there is no $n \geq 0$ and an uncountable 1-1 h such that $\mathrm{Dom}(h) \subseteq X$, $\mathrm{Rng}(h) \subseteq A^n$, every two distinct elements in $\mathrm{Rng}(h)$ are disjoint and whenever $x, y \in \mathrm{Dom}(h)$ and $f(x, y) = 1$, $\{h(x), h(y)\}$ is not OP. (Note that for $n = 0$ this means that there are no uncountable 0-colored homogeneous sets.)

Let M be a model including enough set theory and including X, f, A. Let

$$P=\{\sigma \in P_{\aleph_0}(X) \in \sigma \text{ is homogeneous of color 1 and is } C_M\text{-separated}\}.$$

Clearly by the proof of 1.1, $\Vdash_P$ "X contains an uncountable homogeneous subset". Suppose by contradiction there is $p\in P$ such that $p\Vdash_P$ "A is not increasing". Let $\tilde{B}$ be a name of a subset of A^n such that p forces that $\tilde{B}$ is a counter-example to the increasingness of A. Let $\{\langle p_\alpha, \boldsymbol{b}^\alpha\rangle \mid \alpha<\aleph_1\}$ be such that for every α, $p_\alpha \geqslant p$, $p_\alpha \Vdash \boldsymbol{b}^\alpha \in \tilde{B}$, and for every $\alpha\neq\beta$, $\boldsymbol{b}^\alpha$, $\boldsymbol{b}^\beta$ are pairwise disjoint. Let $p^\alpha=\{a_0^\alpha,\ldots,a_{m_\alpha-1}^\alpha\}$ where $a_0^\alpha<\cdots<a_{m_\alpha-1}^\alpha$, let $\boldsymbol{a}^\alpha=\langle a_0^\alpha,\ldots,a_{m_\alpha-1}^\alpha\rangle$ and $\boldsymbol{b}^\alpha=\langle b_1^\alpha,\ldots,b_n^\alpha\rangle$. W.l.o.g. (1) for every α, $m_\alpha=m$; (2) $\{p_\alpha \mid \alpha<\aleph_1\}$ is a Δ-system with kernel $\{a_0,\ldots,a_{r-1}\}$ where for every $\alpha<\aleph_1$ and $i\leqslant r-1$, $a_i^\alpha=a_i$ and for every α and β, $p_\alpha\cup p_\beta$ is C_M-separated; for every $0\leqslant i<j\leqslant m$ and $\alpha<\beta<\aleph_1$, $f(a_i^\alpha, a_j^\beta)=1$; and (4) for every α, $b_1^\alpha<\cdots<b_n^\alpha$. Let $D_0=\{\boldsymbol{a}^\alpha{}^\frown\boldsymbol{b}^\alpha \mid \alpha<\aleph_1\}$. Hence $D_0\subseteq X^m\times A^n$, let D be the topological closure of D_0. Let $\gamma<\aleph_1$ be such that D is definable in M by a parameter belonging to $|M_\gamma|$, and let α be such that $p_\alpha\cap|M_\gamma|=\{a_0,\ldots,a_{r-1}\}$ and $\mathrm{Rng}(\boldsymbol{b}^\alpha)\cap|M_\gamma|=\emptyset$.

We shall now duplicate $\boldsymbol{a}^\alpha{}^\frown\boldsymbol{b}^\alpha$. Let $\boldsymbol{a}^\alpha=\boldsymbol{a}=\langle a_0,\ldots,a_{m-1}\rangle$, $\boldsymbol{b}^\alpha=\boldsymbol{b}=\langle b_1,\ldots,b_n\rangle$. Let $E^1,\ldots,E^m$ be the set of all those C_M-slices which intersect $\{a_r,\ldots,a_{m-1}\}\cup\{b_1,\ldots,b_n\}$. We represent $\boldsymbol{a}$ as $\boldsymbol{a}_0{}^\frown\cdots{}^\frown\boldsymbol{a}_k$ and $\boldsymbol{b}$ as $\boldsymbol{b}_1{}^\frown\cdots{}^\frown\boldsymbol{b}_k$ where $\boldsymbol{a}_0=\langle a_0,\ldots,a_{r-1}\rangle$, and for $i>0$, $\boldsymbol{a}_i$, $\boldsymbol{b}_i$ are respectively the subsequences of $\boldsymbol{a}$ and $\boldsymbol{b}$ consisting of those elements which belong to E^i. Note that since p_α is C_M-separated, then for every $i>0$, $\boldsymbol{a}_i$ is either empty or consists of one element.

We define by a downward induction formulas φ_i, $i=k,\ldots,0$, with parameters in $|M_\gamma|$ such that $M\vDash\varphi_i[\boldsymbol{a}_1,\ldots,\boldsymbol{a}_i,\boldsymbol{b}_1,\ldots,\boldsymbol{b}_i]$.

$$\varphi_k\equiv\boldsymbol{a}_0{}^\frown\boldsymbol{x}_1{}^\frown\cdots{}^\frown\boldsymbol{x}_k{}^\frown\boldsymbol{y}_1{}^\frown\cdots{}^\frown\boldsymbol{y}_k\in D.$$

Suppose φ_{i+1} has been defined, and we want to define φ_i. There are two cases: (1) $\boldsymbol{x}_i$ consists of one variable and (2) x_i is an empty sequence. Since $M\vDash\varphi_{i+1}[\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{i+1},\boldsymbol{b}_1,\ldots,\boldsymbol{b}_{i+1}]$, it follows that

$$M\vDash Q\boldsymbol{x}_{i+1},\boldsymbol{y}_{i+1}\,\varphi_{i+1}[\boldsymbol{a}_1,\ldots,\boldsymbol{a}_i,\boldsymbol{x}_{i+1},\boldsymbol{b}_1,\ldots,\boldsymbol{b}_i,\boldsymbol{y}_{i+1}].$$

By (*) in case (1), and by the increasingness of A in case (2), there are $\boldsymbol{c}^l$, $\boldsymbol{d}^l$, $l=1,2$, such that $M\vDash\varphi_{i+1}[\boldsymbol{a}_1,\ldots,\boldsymbol{a}_i,\boldsymbol{c}^l,\boldsymbol{b}_1,\ldots,\boldsymbol{b}_i,\boldsymbol{d}^l]$, $\{\boldsymbol{d}^1,\boldsymbol{d}^2\}$ is OP, and if $\boldsymbol{c}^l=\langle c^l\rangle$, $l=1,2$, then $f(c^1,c^2)=1$. Let V_{i+1}^l be basic open sets in X such that $(c^l\in V_{i+1}^l$ and $l=1,2$, and $f(V_{i+1}^1\times V_{i+1}^2)=\{1\}$, and let W^l, $l=1,2$, be a product of basic open sets in A such that $\boldsymbol{d}^l\in W^l$, $l=1,2$, and for every $\boldsymbol{e}^1\in W^1$, $\boldsymbol{e}^2\in W^2$, $\{\boldsymbol{e}^1,\boldsymbol{e}^2\}$ is OP. Let

$$\varphi_1\equiv\exists\boldsymbol{x}_{i+1}^1,\boldsymbol{x}_{i+1}^2,\boldsymbol{y}_{i+1}^1,\boldsymbol{y}_{i+1}^2\left(\bigwedge_{l=1}^{2}\varphi_{i+1}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_i,\boldsymbol{x}_{i+1}^l,\boldsymbol{y}_1,\ldots,\boldsymbol{y}_i,\boldsymbol{y}_{i+1}^l)\right.$$
$$\left.\wedge(\{y_{i+1}^1,y_{i+1}^2\}\text{ is OP})\wedge f(x_{i+1}^1,x_{i+1}^2)=1\right).$$

The last conjunct is added only in case 1.

Starting now from φ_0 and using successively $\varphi_1, \ldots, \varphi_k$ we can construct $\boldsymbol{c}^l = \langle c^l, \ldots, c^l_{m-1}\rangle$, and $\boldsymbol{d}^l$, $l = 1, 2$, such that: $\langle a_0, \ldots, a_{r-1}\rangle \wedge \boldsymbol{c}^l \wedge \boldsymbol{d}^l \in D$, $\{\boldsymbol{d}^1, \boldsymbol{d}^2\}$ is OP, and for every $i = 0, \ldots, m-1$, $f(c^1_i, c^2_i) = 1$. Since D_0 is dense in D, there are $\beta^1, \beta^2 < \aleph_1$ such that $\boldsymbol{a}^{\beta^l} \wedge \boldsymbol{b}^{\beta^l}$ is close enough to $\langle a_0, \ldots, a_{r-1}\rangle \wedge \boldsymbol{c}^l \wedge \boldsymbol{d}^l$, $l = 1, 2$; but then $p_{\beta^1} \cup p_{\beta^2} \in P$ and $\{\boldsymbol{b}^{\beta^1}, \boldsymbol{b}^{\beta^2}\}$ is OP. A contradiction and hence P is as desired.

So far we have dealt with the case when (*) holds. Consider now the case when $\neg$(*) holds. So, there is a sequence $\{\langle a_\alpha, \boldsymbol{b}^\alpha\rangle \mid \alpha < \aleph_1\}$ such that the a_α's are distinct and belong to X, the $\boldsymbol{b}^\alpha$ belong to A^n and they are pairwise disjoint, and whenever $f(a_\alpha, a_\beta) = 1$, $\{\boldsymbol{b}^\alpha, \boldsymbol{b}^\beta\}$ is not OP. If $n = 0$, then $\{a_\alpha \mid \alpha < \aleph_1\}$ is already an uncountable homogeneous set, so P can be chosen to be the trivial forcing. Suppose $n > 0$. We color distinct $\boldsymbol{b}^\alpha, \boldsymbol{b}^\beta$ in two colors according to whether $\{\boldsymbol{b}^\alpha, \boldsymbol{b}^\beta\}$ is OP or not. This is an open coloring hence by Lemma 3.2 there is a c.c.c. forcing set P of power $\aleph_1$ which does not destroy the increasingness of A and decomposes $\{\boldsymbol{b}^\alpha \mid \alpha < \aleph_1\}$ into countably many homogeneous sets. We show that P adds an uncountable homogeneous set to X. Let $\{\boldsymbol{b}^\alpha \mid \alpha \in \Gamma\} = B$ be an uncountable homogeneous set added by P. Since P did not destroy the increasingness of A, for every $\alpha, \beta \in \Gamma$, $\{\boldsymbol{b}^\alpha, \boldsymbol{b}^\beta\}$ is OP, hence $f(a_\alpha, a_\beta) = 0$, and hence $\{a_\alpha \mid \alpha \in \Gamma\}$ is f-homogeneous of color 0. We have thus proved Lemma 3.3, and since we skip the details of the iteration this concludes the proof of Theorem 3.1. □

Question 3.4. Can SOCA be replaced by SOCA1 in Theorem 3.1?

In the remainder of this section we try to generalize OCA to colorings of n-tuples rather than just colorings of pairs. Example 1.7 shows that the most direct generalization of OCA is inconsistent. However, the following axiom generalizing OCA might still be consistent with ZFC.

Axiom OCA(m, k). If X is a second countable Hausdorff space of power $\aleph_1$, and $\mathcal{U}$ is a finite open cover of X^m, then X can be partitioned into $\{X_i \mid i \in \omega\}$ such that for every $i \in \omega$, $(X_i)^m$ intersects at most k elements of $\mathcal{U}$.

Question 3.5. Is it true that for every m there exists a k such that OCA(m, k) is consistent?

In fact we do not even know the answer to the following weakened version of the above question. Is there k such that the following axiom is consistent: "If X is a Hausdorff second countable space and $\mathcal{U}$ is a finite open cover of X^3, then there is an uncountable $A \subseteq X$ such that A^3 intersects at most k elements of $\mathcal{U}$".

At this point it is worthwhile to mention the following theorem of A. Blass [4]. If $\mathcal{U}$ is a symmetric partition of the n-tuples of ${}^\omega 2$ into finitely many open sets, then ${}^\omega 2$ contains a perfect subset in which at most $(n-1)!$ colors appear.

We will prove a weaker generalization of OCA; however rather than formulating this new axiom in topological terms, we translate it into an equivalent statement on colorings of the binary tree.

We first introduce some terminology. Let $\bar{T}=\langle {}^{\omega\geq}2, \leq\rangle$ be the tree of binary sequences of length $\leq\omega$; let $T={}^{\omega>}2$ and $L={}^{\omega}2$. L is regarded as the set of branches of T. For $\nu, \eta \in \bar{T}$, $\nu<\eta$ denotes that ν is a proper initial segment of η, $\nu\wedge\eta$ denotes the maximal common initial segment of ν and η, Λ denotes the empty sequence, if $\nu\leq\eta$, then $[\nu, \eta]$, (ν, η) denote respectively the closed and open intervals with endopoints ν and η, and $[\nu]$, $[\nu)$ denote respectively $[\Lambda, \nu]$ and $[\Lambda, \nu)$. If $A\subseteq L$ let $T[A]=\{\nu\wedge\eta \mid \nu, \eta\in A$ and $\nu\neq\eta\}$; note that $T[A]$ is closed under $\wedge$. For $B\subseteq T$ let $B^{[m]}=\{\sigma\subseteq B \mid |\sigma|=m$ and σ is closed under $\wedge\}$. Let $\nu<_L\eta$ denote that $\nu^\frown\langle 0\rangle\leq\eta$ and $\nu<_R\eta$ mean that $\nu^\frown\langle 1\rangle\leq\eta$. If $\sigma, \tau\in T^{[m]}$ then $\sigma\sim\tau$ means that $\langle\sigma, <_L, <_R\rangle\cong\langle\tau, <_L, <_R\rangle$. A function $f: T^{[m]}\to n$ is called an m-coloring of T; $B\subseteq T$ is f-homogeneous if for every $\sigma, \tau\in B^{[m]}$ such that $\sigma\sim\tau: f(\sigma)=f(\tau)$; $A\subseteq L$ is f-homogeneous if $T[A]$ is.

Let the tree m-coloring axiom be as follows.

Axiom TCA*m*. For every $A\subseteq L$ of power $\aleph_1$ and for every m-coloring f of T, A can be partitioned into countably many f-homogeneous subsets.

Let $\mathrm{TCA}=\bigwedge_{m\in\omega}\mathrm{TCA}m$.

We shall later present a toplogical formulation equivalent to TCA. For the time being the reader can check the following proposition.

Proposition 3.6. (a) OCA$\Rightarrow$TCA1.

(b) $\mathrm{MA}_{\aleph_1}$+TCA1$\Rightarrow$OCA.

Our next goal is the following theorem.

Theorem 3.7. TCA+MA *is consistent.*

Lemma 3.8 (CH). *Let $A\subseteq L$ be of power $\aleph_1$, and let $\mathscr{D}=\{D_i \mid i\in\omega\}$ be a partition of the levels of T into finite intervals, that is, D_i can be written as $[n_i, n_{i+1})$ where $n_0=0$ and $n_i<n_{i+1}$. Then there is a c.c.c. forcing set $P=P_{A,\mathscr{D}}$ of power $\aleph_1$ such that after forcing with P, A can be partitioned into countably many sets $\{A_j \mid j\in\omega\}$ such that for every j, $T[A_j]$ intersects each D_i in at most one point.*

Proof. Let M be a model with universe $\aleph_1$ which encodes T, A and $\mathscr{D}$, and let $C=C_M$. P will consist of all finite approximations of the desired partition $\{A_j \mid j\in\omega\}$ in which each A_j intersects each C-slice in at most one point. To be more precise let $\{\alpha_i \mid i<\aleph_1\}$ be an order preserving enumeration of C, let $E_i=[\alpha_i, \alpha_{i+1})$, and let $\mathscr{E}=\{E_i \mid i<\aleph_1\}$. $\mathscr{E}$ is called the set of C-slices. $P=\{f \mid \mathrm{Dom}(f)\in P_{\aleph_0}(A)$, $\mathrm{Rng}(f)\subseteq\omega$ and for every $j\in\omega$: for every D_i, $|T[f^{-1}(j)]\cap D_i|\leq 1$ and for every E_i, $|f^{-1}(j)\cap E_i|\leq 1\}$.

By the standard duplication method one can easily show that P is c.c.c., and clearly $\Vdash_P$ "A can be partitioned into $\{A_j \mid j \in \omega\}$ such that for every j and i, $|T[A_j] \cap D_i| \leq 1$". $\square$

Let $\sigma \in \bigcup_{m \in \omega} T^{[m]}$ and $\nu \in T$, $\sigma < \nu$ denotes that $\max(\{\nu \wedge \xi \mid \xi \in \sigma\}) \in \sigma$. Note that (1) if $\sigma < \nu$, then $\sigma \cup \{\nu\} \in \bigcup_{m \in \omega} T^{[m]}$ and (2) σ can be written as $\{\xi_1, \ldots, \xi_m\}$ where for each $i < m$, $\{\xi_1, \ldots, \xi_i\} \in T^{[i]}$ and $\{\xi_1, \ldots, \xi_i\} < \xi_{i+1}$. Let $\sigma \in T^{[m]}$ and $\nu, \eta \in T$; $\nu \sim_\sigma \eta$ if there is an isomorphism between $\langle \sigma \cup \{\nu\}, <_L, <_R \rangle$ and $\langle \sigma \cup \{\eta\}, <_L, <_R \rangle$ which is the identity on σ. For $\nu \in T$ let $\mathrm{lth}(\nu)$ be the length of ν. Let $\boldsymbol{n} = \{n_i \mid i \in \omega\}$ be a strictly increasing sequence of natural numbers, let $\sigma \subseteq T$ and $\nu \in T$; we say that σ, ν are $\boldsymbol{n}$-separated if for some i: for every $\eta \in \sigma$, $\mathrm{lth}(\eta) < n_i$ and $\mathrm{lth}(\nu) \geq n_i$.

Lemma 3.9 (CH). *Let $f : T^{[m+1]} \to n$ be an $m+1$-coloring of T and $A \subseteq L$ be uncountable. Then there is a c.c.c. forcing set $P = P^1_{A,f}$ of power $\aleph_1$ such that after forcing with P we have the following situation: there is a strictly increasing sequence $\boldsymbol{n}$ with $n_0 = 0$ and an uncountable $B \subseteq A$ such that for every $\sigma \in T[B]^{[m]}$ and $\nu, \eta \in T[B]$: if $\sigma < \nu$, η; $\nu \sim_\sigma \eta$ and σ, ν are $\boldsymbol{n}$-separated, then $f(\sigma \cup \{\nu\}) = f(\sigma \cup \{\eta\})$.* (We call such B a prehomogeneous set.)

Moreover, (*) there is a countable $A' \subseteq A$ such that for every $a \in A - A'$ there is $p \in P$ such that $p \Vdash_P a \in B$.

Before proving Lemma 3.9, let us see how Theorem 3.7 follows from Lemmas 3.8 and 3.9.

Proof of Theorem 3.7. As usual we deal just with the atomic step in the iteration. So, given a subset $A \subseteq L$ of power $\aleph_1$ and an m-coloring f of T we have to find a c.c.c. forcing set of power $\aleph_1$ such that after forcing with it A can be partitioned into countably many f-homogeneous subsets. We prove this by induction on m.

The case $m = 1$ follows from the proof of the consistency of OCA.

Suppose by induction for every m-coloring f of T and every $A \subseteq L$ of power $\aleph_1$ there is a c.c.c. forcing set $P = P_{A,f}$ of power $\aleph_1$ such that $\Vdash_P$ "A can be partitioned into countably many f-homogeneous subsets".

Let V be a universe satisfying CH, $A \subseteq L$ be of power $\aleph_1$ and f be an $m+1$-coloring of T. Let $\{Q_i \mid i \in \omega\}$, $\{P_i \mid i \in \omega\}$ be a finite support iteration, Q_0 is trivial, P_i is a Q_i-name for the forcing set $P^1_{A,f}$ from Lemma 3.9 in the universe V^{Q_i}, and $Q_{i+1} = Q_i * P_i$. Let $P^2_{A,f} = \bigcup_{i \in \omega} Q_i$. We denote $P^2_{A,f}$ by Q. In V^Q we have a family $\{B_i \mid i \in \omega\}$ of prehomogeneous subsets of A, and corresponding to each B_i we have a sequence $\boldsymbol{n}^i$. By (*) of 3.9 it is easy to check that $|A - \bigcup_{i \in \omega} B_i| \leq \aleph_0$. Let $D^i_j = \{\nu \in T \mid n^i_j \leq \mathrm{lth}(\nu) < n^i_{j+1}\}$ and $\mathscr{D}^i = \{D^i_j \mid j \in \omega\}$. Let R be the Q-name of the following forcing set. R is gotten by a finite support iteration of $P_{A,\mathscr{D}^i}$ of Lemma 3.8. After forcing with R each B_i is partitioned into countably many sets which we denote by $\{B_{ij} \mid j \in \omega\}$. It is easy to see that for every B_{ij}: if $\sigma \in T[B_{ij}]^{[m]}$, $\nu, \eta \in T[B_{ij}]$, $\sigma < \nu, \eta$ and $\nu \sim_\sigma \eta$, then $f(\sigma \cup \{\nu\}) = f(\sigma \cup \{\eta\})$.

We can now define an m-coloring on each $T[B_{ij}]$. The color $f_{ij}(\sigma)$ where $\sigma \in T[B_{ij}]^{[m]}$ is the sequence of colors of the form $f(\sigma \cup \{\nu\})$ where the ν's belong to $T[B_{ij}]$ and they represent all equivalence classes of $\sim_\sigma$ in which $\sigma < \nu$. More precisely for every $\sigma \in T[B_{ij}]^{[m]}$ let $\nu_1^\sigma, \ldots, \nu_{k_\sigma}^\sigma > \sigma$ be such that for every $\nu > \sigma$ there is a unique i such that $\nu \sim_\sigma \nu_i^\sigma$. Moreover we pick the ν_i^σ's in such a way that if $\tau \sim \nu$, then for every i there is an isomorphism between $\langle \tau \cup \{\nu_i^\tau\}, <_L, <_R \rangle$ and $\langle \sigma \cup \{\nu_i^\sigma\}, <_L, <_R \rangle$ which maps τ onto σ. We define $f_{ij}(\sigma) = \langle f(\sigma \cup \{\nu_1^\sigma\}), \ldots, f(\sigma \cup \{\nu_{k_\sigma}^\sigma\}) \rangle$.

By the induction hypothesis there is a c.c.c. forcing set S of power $\aleph_1$ such that $\Vdash_S$ "Each B_{ij} can be partitioned into countably many f_{ij}-homogeneous sets". It is easy to see that if $B \subseteq B_{ij}$ is f_{ij}-homogeneous, then B is f-homogeneous. Hence after forcing with $Q * R * S$, A can be partitioned into countably many f-homogeneous subsets. This completes the proof of Theorem 3.7. □

Proof of Lemma 3.9. For $B \subseteq T$ and $a \in T$, let $B^{[m,a]} = \{\sigma \in B^{[m]} \mid \text{there is } \nu \in \sigma \text{ such that } \nu < a\}$. Let $f: T^{[m+1]} \to n$ and A be as in 3.9, let M be a model with universe $\aleph_1$ which has T, f and A as predicates. Let $\{\alpha_i \mid i < \aleph_1\}$ be an order preserving enumeration of C_M, $M_i = M \restriction \alpha_i$, $E_i = [\alpha_i, \alpha_{i+1})$ and $A'' = A \cap [\alpha_0, \aleph_1)$. For every $a \in A''$ we define a coloring $f_a : T^{[m,a]} \to n$. Suppose $a \in E_i$, for every finite subset $C \subseteq T$ there is a function $g_C : C^{[m,a]} \to n$ such that for every formula $\varphi(x)$ in the language of M, and with parameters from M_i: if $M \vDash \varphi[a]$, then for every $\alpha < \aleph_1$ there are $b, c \in A$ such that $b, c > \alpha$, $M \vDash \varphi[b] \wedge \varphi[c]$ and for every $\sigma \in C^{[m,a]}$, $\sigma < b \wedge c$ and $f(\sigma \cup \{b \wedge c\}) = g_C(\sigma)$. The existence of such g_C is proved, as in the analogous argument in the proof of the consistency of OCA. By Konig's lemma we can choose the g_C's to be pairwise compatible. Let $f_a = \bigcup \{g_C \mid C \in P_{\aleph_0}(T)\}$.

We are ready to define the forcing set $P = P^1_{A,f}$ of Lemma 3.9. An element p of P is an object of the form $\langle \boldsymbol{n}, C \rangle$ where $\boldsymbol{n} = \langle n_0, \ldots, n_{k-1} \rangle$ is a strictly increasing finite sequence of natural numbers with $n_0 = 0$, C is a finite C_M-separated subset of A'' and the following conditions hold: (1) for every distinct $a, b \in C$, $\operatorname{lth}(a \wedge b) < n_{k-1}$, denote by $n_{a,b}$ the maximal n_i such that $n_i \leq \operatorname{lth}(a \wedge b)$; (2) let $f_a \restriction k$ abbreviate $f_a \restriction \{\nu \in T \mid \operatorname{lth}(\nu) < k\}^{[m,a]}$, then for every distinct $a, b \in C$, $f_a \restriction n_{a,b} = f_b \restriction n_{a,b}$; and (3) for every distinct $a, b \in C$ and for every $\sigma \in \operatorname{Dom}(f_a \restriction n_{a,b})$, $f(\sigma \cup \{a \wedge b\}) = f_a(\sigma)$. We denote $\boldsymbol{n} = \boldsymbol{n}^p$, $n_i = n_i^p$, $C = C_p$ and $n_{a,b} = n_{a,b}^p$.

Let $p, q \in P$, then $p \leq q$ if $\boldsymbol{n}^p \leq \boldsymbol{n}^q$ and $C_p \subseteq C_q$.

We prove that P is c.c.c. Let $\{p_\alpha \in \alpha < \aleph_1\} \subseteq P$. W.l.o.g. (1) for every $\alpha, \beta < \aleph_1$, $\boldsymbol{n}^{p_\alpha} = \boldsymbol{n}^{p_\beta} = \boldsymbol{n} = \langle n_0, \ldots, n_{k-1} \rangle$, and $\{C_{p_\alpha} \mid \alpha < \aleph_1\}$ is a Δ-system; (2) for every $\alpha < \beta < \aleph_1$, $C_{p_\alpha} = \{a_{\alpha,0}, \ldots, a_{\alpha,r-1}, a_{\alpha,r}, \ldots, a_{\alpha,s-1}\}$ where $\{a_{\alpha,0}, \ldots, a_{\alpha,r-1}\}$ is the kernel of the Δ-system, and $a_{\alpha,0} < \cdots < a_{\alpha,s-1} < a_{\beta,r}$; (3) for every $\alpha, \beta < \aleph_1$ and $i < s$, $a_{\alpha,i} \restriction n_{k-1} = a_{\beta,i} \restriction n_{k-1}$ and $f_{a_{\alpha,i}} \restriction n_{k-1} = f_{a_{\beta,i}} \restriction n_{k-1}$.

We regard each C_{p_α} as an element of L^s. We use the usual topology on L and define D to be the topological closure of $\{C_{p_\alpha} \mid \alpha < \aleph_1\}$ in L^s. D is definable from some parameter e in M. Let i_0 be such that $e, a_{\alpha,0}, \ldots, a_{\alpha,r-1} < \alpha_{i_0}$, and let p_α be such that $\alpha_{i_0} \leq a_{\alpha,r}, \ldots, a_{\alpha,s-1}$.

We apply the duplication argument to p_α. Let us denote $p_\alpha = p$, $C_p = C$ and $a_{\alpha,i} = a_i$. We define by a downward induction formulas $\varphi_{s-1}(x_r, \ldots, x_{s-1}), \ldots, \varphi_r(x_r), \varphi_{r-1}$ with parameters $<\alpha_{i_0}$ such that $M \vDash \varphi_i[a_r, \ldots, a_i]$. $\varphi_{s-1}(x_r, \ldots, x_{s-1})$ is the formula saying that $\langle a_0, \ldots, a_{r-1}, x_r, \ldots, x_{s-1}\rangle \in D$. Suppose φ_{i+1} has been defined. By the definition of f_a there are b^1, b^2 such that $M \vDash \varphi_{i+1}[a_r, \ldots, a_i, b^j]$, $j = 1, 2$, $\mathrm{lth}(b^1 \wedge b^2) \geqslant n_{k-1}$ and for every $\sigma \in \mathrm{Dom}(f_{a_{i+1}} \restriction n_{k-1})$, $f(\sigma \cup \{b^1 \wedge b^2\}) = f_{a_{i+1}}(\sigma)$. Let $\nu_{i+1} = b^1 \wedge b^2$ and

$$\varphi_i(x_r, \ldots, x_i) \equiv \exists y^1, y^2 \left(\bigwedge_{j=1}^{2} \varphi_{i+1}(x_1, \ldots, x_i, y^j) \wedge (y^1 \wedge y^2 = \nu_{i+1}) \right).$$

Next we construct by induction sequences $\langle b_r^j, \ldots, b_{s-1}^j\rangle = \boldsymbol{b}^j$, $j = 1, 2$ such that: $\langle a_0, \ldots, a_{r-1}\rangle \frown \boldsymbol{b}^j \in D$, and for every $i = r, \ldots, s-1$, $b_i^1 \wedge b_i^2 = \nu_i$. Since $\{C_{p_\alpha} \mid \alpha < \aleph_1\}$ is dense in D there are $\alpha, \beta < \aleph_1$ such that for every $i = r, \ldots, s-1$, $a_{\alpha,i} \wedge a_{\beta,i} = \nu_i$. Let $n_k > \max(\{\mathrm{lth}(\nu_i) \mid i = r, \ldots, s-1\})$; recalling that for every i, $f_{a_{\alpha,i}} \restriction n_{k-1} = f_{a_{\beta,i}} \restriction n_{k-1}$, it is easy to see that $\langle \boldsymbol{n} \frown \langle n_k\rangle, C_{P_\alpha} \cup C_{P_\beta}\rangle \in P$. Hence P is c.c.c.

P is not yet as required in Lemma 3.9, since if G is P-generic, $\bigcup\{C_p \mid p \in G\}$ need not be uncountable. However, by a standard argument, it is easy to find a countable set $A' \subseteq A$ such that if $P' = \{p \in P \mid C_p \cap A' = \emptyset\}$, then for every P'-generic filter G $\bigcup\{C_p \mid p \in G\}$ is uncountable. P' is obviously as required in 3.9. □

This concludes the proof of Theorem 3.7.

Remark. As in Lemma 3.2 we can also prove that TCA + MA + ISA is consistent.

Question 3.10. Prove that TCA$m \not\Rightarrow$ TCA$m+1$.

Our next goal is to find a generalization of OCA which is equivalent to TCA.

Let $D_m(A)$ be the set of 1-1 m-tuples from A. Let X be second countable and of power $\aleph_1$. An open m-coloring of X is a finite open cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ of $D_m(X)$ such that each U_i is symmetric. We next define what it means for an open m-coloring to be strongly open. We define by a downward induction $\varphi_i(V_1, \ldots, V_i)$ where φ_i is a property of i-tuples of open sets from X. $\varphi_m(V_1, \ldots, V_m) \equiv \exists U_i\ (V_1 \times \cdots \times V_m \subseteq U_i)$. Suppose φ_{i+1} has been defined;

$$\varphi_i'(V_1, \ldots, V_i) \equiv (\forall x_1, x_2 \in V_i)\, (x_1 \neq x_2 \rightarrow (\exists V^1, V^2)\, (x_1 \in V^1 \wedge x_2 \in V^2 \wedge (V^1, V^2 \text{ are open}) \wedge \varphi_{i+1}(V_1, \ldots, V_{i-1}, V^1, V^2))),$$

$$\varphi_i(V_1, \ldots, V_i) \equiv \bigwedge \{\varphi_i'(V_{\pi(1)}, \ldots, V_{\pi(i)}) \mid \pi \text{ is a permutation of } i\}.$$

Definition 3.11. Let $\mathcal{U}$ be an open m-coloring of X. $\mathcal{U}$ is *strongly open* if $\varphi_1(X)$ holds.

Let t_m be the number of isomorphism types of models of the form $\langle \sigma, <_L, <_R\rangle$

where $\sigma \in T^{[m]}$. Let $\mathcal{U}$ be an m-coloring of X. $A \subseteq X$ is $\mathcal{U}$-homogeneous if there is a subset $\mathcal{U}' \subseteq \mathcal{U}$ such that $|\mathcal{U}'| \leq t_m$ and $D_m(A) \subseteq \bigcup\{U \mid U \in \mathcal{U}'\}$.

Axiom OCA*m*. If X is second countable and of power $\aleph_1$ and $\mathcal{U}$ is a strongly open m-coloring of X, then X can be partitioned into countably many $\mathcal{U}$-homogeneous subsets.

Theorem 3.12. (a) OCA$m+1 \Rightarrow$TCAm.
(b) TCA$m + MA_{\aleph_1} \Rightarrow$ OCA$m+1$.

Proof. (a) Assume OCA$m+1$, and let $f: T^{[m]} \rightarrow n$ be an m-coloring of T and $A \subseteq L$ be of power $\aleph_1$. For every $i \in N$ and $\sigma \in T^{[m]}$ we define a symmetric open subset of $D_{m+1}(A)$:

$$U_{\sigma,i} = \{\langle a_0, \ldots, a_m\rangle \in D_m(A) \mid \langle T[\{a_0, \ldots, a_m\}], <_L, <_R\rangle \cong \langle \sigma, <_L, <_R\rangle \text{ and } f(T[\{a_0, \ldots, a_m\}] = i)\}.$$

Clearly $\mathcal{U} \overset{\text{def}}{=} \{U_{\sigma,i} \mid \sigma \in T^{[m]} \text{ and } i \in n\}$ is a finite open cover of $D_{m+1}(A)$, and it is easy to check that $\mathcal{U}$ is a strongly open $(m+1)$-coloring of A. Applying OCA$m+1$ to A and $\mathcal{U}$ one gets a countable partition of A into $\mathcal{U}$-homogeneous subsets. It is easy to check that these sets are in fact f-homogeneous.

(b) Assume $\text{MA}_{\aleph_1}$+TCAm. Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a strongly open $(m+1)$-coloring of a second countable space X of power $\aleph_1$. W.l.o.g. X is Hausdorff. Let $\mathcal{B}$ be a countable open base of X. A tree approximation of $\mathcal{U}$ is a function g such that: $\text{Dom}(g) \subseteq T$, $\text{Rng}(g) \subseteq \mathcal{B}$ and (1) if $\eta < \nu \in \text{Dom}(g)$, then $\eta \in \text{Dom}(g)$; (2) let $\nu, \eta \in \text{Dom}(g)$; then if ν and η are incomparable with respect to $\leqslant$, then $g(\nu) \cap g(\eta) = \emptyset$, and if $\nu \leqslant \eta$, then $g(\nu) \supseteq g(\eta)$; and (3) if $i \leqslant m$ and $\nu_0, \ldots, \nu_i \in \text{Dom}(g)$ are incomparable in T, then $\varphi_{i+1}(g(\nu_0), \ldots, g(\nu_i))$ holds. Let g be an approximation of $\mathcal{U}$, and let $B \subseteq X$; we say that g is an approximation of $\mathcal{U}$ on B if: (1) for every $b \in B$ there is a branch t_b of Dom(g) such that $b \in \bigcap\{g(\nu) \mid \nu \in t_b\}$; and (2) the function mapping b to t_b is 1-1.

Using $\text{MA}_{\aleph_1}$ it is easy to see that there is a family $\{\langle g_i, B_i\rangle \mid i \in \omega\}$ such that g_i is an approximation of $\mathcal{U}$ on B_i and $\bigcup_{i\in\omega} B_i = X$. W.l.o.g. $\text{Dom}(g_i) = T$. Let $A_i = \{t_b \mid b \in B_i\}$, hence $A_i \subseteq L$. For every i we now define an m-coloring f_i of T. Let $\sigma \in T^{[m]}$; if there is no $C \subseteq A_i$ such that $T[C] = \sigma$, define $f_i(\sigma) = 0$; otherwise, let $t_{b_0}, \ldots, t_{b_m}$ be such that $T[\{t_{b_0}, \ldots, t_{b_m}\}] = \sigma$ and let $f(\sigma) = i$ where $\langle b_0, \ldots, b_m\rangle \in U_i$. Clearly, $f(\sigma)$ does not depend on the choice of $b_0, \ldots, b_m$. One can easily check that if $A \subseteq A_i$ is f_i-homogeneous, then $\{b \mid t_b \in A\}$ is $\mathcal{U}$-homogeneous, hence OCA$m+1$ follows. □

Remarks. (a) We did not mention the polarized versions of TCA, however the proof that they are consistent resembles the proof that TCA is consistent.

(b) The consistency of OCAm or TCA implies by absoluteness a special case of Blass' theorem [4], namely that if $\mathcal{U}$ is a strongly open coloring of ${}^{\omega}2$, then ${}^{\omega}2$

contains a perfect set in which at most t_m colors appear. The existence of such a perfect set is a Σ_1^1 statement, and since it holds in some extension it must exist in the ground model.

The main question in this matter is whether our consistency result can be strengthened to include all open colorings as in Blass' theorem.

An open coloring can be regarded as a continuous function from $X \times X$ to the set of colors equipped with its descrete topology. It seems thus natural to examine partition theorems for general continuous functions. We did not investigate these questions thoroughly, however here is one example of such a theorem.

Let the nowhere denseness axiom be as follows

Axiom NWDA2. If X and Y are second countable Hausdorff spaces, $|X| = \aleph_1$ and Y is regular and does not contain isolated points, and if $f: D_2(X) \to Y$ is a symmetric continuous function, then X can be partitioned into $\{A_i \mid i \in \omega\}$ such that for every $i, j \in \omega$, $f(A_i \times A_j)$ is nowhere dense.

Note that even the weakest form of NWDA does not follow from ZFC, for if $A \subseteq \mathbb{R}$ is an uncountable Lusin set (i.e. its intersection with every nowhere dense set is countable) and $f(a, b) = a + b$, then for every uncountable $B \subseteq A$, $f(B \times B)$ if of the second cateogry.

Question 3.13. Does NWDA2 follow from $MA_{\aleph_1}$?

Theorem 3.14. MA+NWDA2 *is consistent.*

Proof. We deal with the atomic step in the interation, and we assume CH in every intermediate stage. Let X, Y, f be as in the axiom, let $\mathcal{B}$ and $\mathcal{C}$ be countable bases of X and Y respectively, and let M be a model whose universe is $\aleph_1$ and which encodes f, X, Y, $\mathcal{B}$ and $\mathcal{C}$. Let $\{E_i \mid i < \aleph_1\}$ be an enumeration of the C_M-slices in an increasing order. Let $U \subseteq Y$ be a finite union of elements of $\mathcal{C}$, and let $a \in E_\alpha$. We say that U is permissible for a, if for every formula $\varphi(x)$ with parameters in $\bigcup_{\beta<\alpha} E_\beta$: if $M \models \varphi[a]$, then there are distinct b, c such that $M \models \varphi[b]$, $M \models \varphi[c]$ and $f(b, c) \notin \mathrm{cl}(U)$. Let

$$P = P^1_{X,Y,f} = \{\langle \sigma, U \rangle \mid \sigma \in P_{\aleph_0}(X),\ \sigma \text{ is } C_M\text{-separated},$$
$$f(D_2(\sigma)) \cap \mathrm{cl}(U) = \emptyset, \text{ and for every } a \in \sigma,\ U \text{ is permissible for } a\}.$$

$\langle \sigma_1, U_1 \rangle \leq \langle \sigma_2, U_2 \rangle$ if $\sigma_1 \subseteq \sigma_2$ and $U_1 \subseteq U_2$.

One can easily check that if U is permissible for a and $V \subseteq Y$ is open and non-empty, then there is $U_1 \supseteq U$ such that $U_1 \cap V \neq \emptyset$ and Y_1 is permissible for a. It is easy to check by the duplication argument that P is c.c.c. Let G be P-generic and $A = \bigcup \{\sigma \mid \exists U\, (\langle \sigma, U \rangle \in G\}$. Then $f(D_2(A))$ is nowhere dense. Let $P^2_{X,Y,f}$ be the forcing set gotten by iterating $P^1_{X,Y,f}$ ω times with finite support. It is easy to check that if G is $P^2_{X,Y,f}$ generic, then in $V[G]$, X has a partition $\{A_i \mid i \in \omega\}$ such that for every $i \in \omega$, $f(D_2(A_i))$ is nowhere dense.

The proof will be completed if we show the following claim. □

Claim. *Let $\{A_i \mid i \in \omega\}$ be a family of second countable spaces of power $\aleph_1$, Y be a second countable space without isolated points, and for every $i \leqslant j$ let $f_{ij}: A_i \times A_j \to Y$ be a continuous function. Then there is a c.c.c. forcing set P of power $\aleph_1$ such that after forcing with P each A_i can be partitioned into $\{A_{ij} \mid j \in \omega\}$ such that for every distinct $\langle i_1, j_1\rangle$, $\langle i_2, j_2\rangle$, $f_{i_1 i_2}(A_{i_1 j_1} \times A_{i_2 j_2})$ is nowhere dense.*

We leave it to the reader to construct such P. (Here one does not have to assume CH in the ground model.)

Question 3.15. Let NWDAm denote the axiom analogous to NWDA2 where m-place functions replace 2-place functions. Prove that NWDAm is consistent.

4. The semi open coloring axiom does not imply the open coloring axiom; the tail method

In this section we present another trick called the "tail method". This method is used in the proof of the following theorem.

Theorem 4.1. MA + SOCA + ¬OCA + $2^{\aleph_0} = \aleph_2$ *is consistent.*

Indeed in Section 11 we prove that $\text{MA}_{\aleph_1}$ + OCA ⇒ $2^{\aleph_0} = \aleph_2$ and in Section 5 we prove that $\text{MA}_{\aleph_1}$ + SOCA + $2^{\aleph_0} > \aleph_2$ is consistent, hence this means that $\text{MA}_{\aleph_1}$ + SOCA + ¬OCA + $2^{\aleph_0} > \aleph_2$ is consistent.

Still Theorem 4.1 adds some information, but more importantly it is a simple application of the tail method and thus will well serve in presenting this method.

The consistency of $\text{MA}_{\aleph_1}$ with the existence of an entangled set which is proved in [1], implies that $\text{MA}_{\aleph_1}$ + ¬SOCA + ¬OCA is consistent.

However we were unsuccessful in proving or disproving the following.

Question 4.2. Does OCA imply SOCA? Does MA + OCA imply SOCA?

Proof of Theorem 4.1. We give a detailed description of the proof, but skip the details which are standard; we also skip some formalities in order to simplify notations.

Definition. $\langle X, f\rangle$ is a *SOC pair* if f is a SOC of X; it is called an *OC pair* if in addition $f^{-1}(0)$ is open in $D(X)$.

We want to construct a universe W in which MA+SOCA+$\neg$OCA holds. To do this we start with a universe V and an OC pair $\langle Y, g\rangle \in V$ such that $V \models$ "CH+$2^{\aleph_1} = \aleph_2$ and Y does not contain uncountable g-homogeneous subsets". W is gotten from V by a finite support iteration of forcing sets $\{P_\alpha \mid \alpha \leq \aleph_2\}$, and we want that $\langle Y, g\rangle$ will be a counter-example to OCA in W. So we prepare in advance a list of tasks which will enumerate all possible SOC pairs $\langle X, f\rangle$ and all possible c.c.c. forcing sets of power $\aleph_1$. In addition we prepare a 1-1 enumeration $\{y_\beta \mid \beta < \aleph_1\}$ of Y. We define by induction on $\alpha < \aleph_2$ a forcing set P_α and a club $C_\alpha \subseteq \aleph_1$. Let $Y_\alpha = \{y_\beta \mid \beta \in C_\alpha\}$. We call Y_α the αth tail of Y. Our induction hypothesis is that $\Vdash_{P_\alpha} Y_\alpha$ does not contain uncountable g-homogeneous subsets.

Let $P = P_{\aleph_2}$. It is clear that the induction hypothesis assures that $\Vdash_P$ "Y is not a countable union of homogeneous sets".

If δ is a limit ordinal, then $P_\delta = \bigcup_{\alpha<\delta} P_\alpha$. We choose a club $C_\delta \subseteq \aleph_1$ such that for every $\alpha < \delta$, $|C_\delta - C_\alpha| \leq \aleph_0$. We want to check that the induction hypothesis holds.

Case 1. $\mathrm{cf}(\delta) = \aleph_1$. Suppose by contradiction for some P_δ-generic filter G there is $A \in V[G]$ such that A is an uncountable g-homogeneous subset of Y_δ. Since $\langle Y, g\rangle$ is an OC pair we can assume that A is closed; and since Y is second countable there is $\alpha < \delta$ such that $A \in V(G \cap P_\alpha]$. $A \cap Y_\alpha$ is an uncountable homogeneous subset of Y_α belonging to $V[G \cap P_\alpha]$, and this contradicts the induction hypothesis.

Case 2. $\mathrm{cf}(\delta) = \aleph_0$. Suppose by contradiction that there is a P_δ-generic G and $A \in V[G]$ such that A is an uncountable homogeneous subset of Y_δ. Let $\{\alpha_i \mid i \in \omega\}$ be an increasing sequence converging to δ. Then there are A_i, $i \in \omega$, such that $A = \bigcup_{i\in\omega} A_i$ and $A_i \in V[G \cap P_{\alpha_i}]$. Some A_i is uncountable, hence $A_i \cap Y_{\delta_i}$ is uncountable. This again contradicts the induction hypothesis.

Let us see how to define $P_{\alpha+1}$, $C_{\alpha+1}$ in the successor stage. If our αth task is to deal with a c.c.c. forcing set P_α, we will use a version of the explicit contradiction method, this will be explained later.

We first deal with the case when the αth task is a P_α-name of a SOC pair $\langle X_\alpha, f_\alpha\rangle$. For a SOC pair $\langle Z, h\rangle$ such that Z does not contain uncountable 0-colored sets, let $M(Z, h, Y, g) = M$ be a model whose universe is $\aleph_1$, and which encompasses enough set theory, and Z, h, Y and g. Let C_M be the club of initial elementary submodels of M. Let $P(Z, h, Y, g)$ be the forcing set consisting of all C_M-separated finite 1-colored subsets of Z.

Suppose X_α does not contain uncountable 0-colored sets. We want to add to X_α an uncountable 1-colored set without destroying the induction hypothesis. For this we need the following lemma.

Lemma 4.3 (CH). *Let $\langle Y, g\rangle$ be an OC pair which does not contain uncountable homogeneous subsets, and let $\{y_\beta \mid \beta < \aleph_1\}$ be a* 1-1 *enumeration of Y. Let $\langle X, f\rangle$ be a SOC pair which does not contain uncountable* 0*-colored subsets. Then there is a club $C \subseteq \aleph_1$ an uncountable $X' \subseteq X$ and a c.c.c. forcing set $P_{X',f} = P$ of power $\aleph_1$ such that $\Vdash_P$ "$\{y_\beta \mid \beta \in C\}$ does not contain uncountable homogeneous subsets".*

Let $\{\beta_\gamma \mid \gamma<\aleph_1\}$ be an order preserving enumeration of C_α. In order to define $P_{\alpha+1}$, $C_{\alpha+1}$ we apply Lemma 4.3 to $\langle Y_\alpha, g\rangle$, $\langle X, f\rangle$ and the enumeration $\{y_{\beta_\gamma} \mid \gamma<\aleph_1\}$ of Y_α. Let X', C be respectively the subspace of X and the club whose existence is assured in 4.3. We define $P_{\alpha+1}$ to be $P_\alpha * P_{X',f}$ and $C_{\alpha+1}=\{\beta_\gamma \mid \gamma\in C\}$. It is clear that $P_{\alpha+1}$, $C_{\alpha+1}$ satisfy the induction hypothesis.

Lemma 4.3 is broken into two claims.

Lemma 4.4 (CH). *Let $\langle X, f\rangle$ be a SOC pair, $\langle Y, g\rangle$ be an OC pair and $\{y_\beta \mid \beta<\aleph_1\}$ be a 1-1 enumeration of Y. Suppose X does not contain uncountable 0-colored subsets, and Y does not contain uncountable homogeneous subsets, then there are uncountable $X'\subseteq X$ and a club $C\subseteq\aleph_1$ such that letting Y' be $\{y_\beta \mid \beta\in C\}$, for every uncountable 1-1 function $h\subseteq X'\times Y'$ and every $l\in\{0,1\}$, there are $x_1, x_2\in \mathrm{Dom}(h)$ such that $f(x_1,x_2)=1$ and $g(h(x_1),h(x_2))=l$.*

Lemma 4.5 (CH). *Let $\langle X', f\rangle$, $\langle Y', g\rangle$ be as assured by Lemma 4.4, and let $P=P(X', f, Y', g)$, then $\Vdash_P$ "Y' does not contain uncountable homogeneous subsets".*

Proof of Lemma 4.4. We first prove the following claim.

Claim 1. *Let $\langle X, f\rangle$, $\langle Y, g\rangle$ and $\{y_\beta \mid \beta<\aleph_1\}$ be as in 4.4, let $F\subseteq X\times Y$ and $l\in\{0,1\}$, and suppose that for every $x_1, x_2\in X$ and $y_1, y_2\in Y$: if $x_1\neq x_2$, $y_1\neq y_2$, $\langle x_1, y_1\rangle$, $\langle x_2, y_2\rangle\in F$ and $f(x_1,x_2)=1$, then $g(y_1,y_2)=l$. Then there are at most countably many x's in X for which $|\{y \mid \langle x,y\rangle\in F\}|>\aleph_0$, and there are at most countably many y's in Y for which $|\{x \mid \langle x,y\rangle\in F\}|>\aleph_0$.*

Proof. Let $F(x)=\{y \mid \langle x,y\rangle\in F\}$ and $F^{-1}(y)=\{x \mid \langle x,y\rangle\in F\}$. Suppose by contradiction that $A\overset{\text{def}}{=}\{x \mid |F(x)|=\aleph_1\}$ is uncountable. Since Y does not contain uncountable l-colored sets, for every $x\in A$ there are $y_x^1, y_x^2\in F(x)$ such that $g(y_x^1, y_x^2)=1-l$, and the choice of the y_x^i's can be made so that for every $u\neq v$ in A, $\{y_u^1, y_u^2\}\cap\{y_v^1, y_v^2\}=\emptyset$. By the second countability of Y and the openness of g, there is an uncountable $B\subseteq A$ such that for every distinct $u, v\in B$, $g(y^1, y^2)=1-l$. Since X does not contain uncountable 0-colored sets there are $u, v\in B$ such that $f(u,v)=1$. This contradicts the assumption about F, since $f(u,v)=1$, $y_u^1\neq y_u^2$ for $\langle u, y_u^1\rangle$, $\langle v, y_v^2\rangle\in F$ but $g(y_u^1, y_v^2)\neq l$.

The agrument why $|\{y \mid |F^{-1}(y)|>\aleph_0\}|\leq\aleph_0$ is similar. □

We now return to the proof of Lemma 4.4. For F as above let $D(F)=\{x \mid |F(x)|\leq\aleph_0\}$ and $R(F)=\{y \mid |F^{-1}(y)|\leq\aleph_0\}$. Let $\{F_i \mid i<\aleph_1\}$ be an enumeration of all closed subsets $F\subseteq X\times Y$ which satisfy $(*)$: there is $l=l_F\in\{0,1\}$ such that for every $x_1, x_2\in X$ and $y_1, y_2\in Y$: if $f(x_1,x_2)=1$, $y_1\neq y_2$ and $\langle x_1,y_1\rangle$, $\langle x_2,y_2\rangle\in F$, then $g(y_1,y_2)=l$.

We define by induction on $i<\aleph_1$, $x_i\in X$ and $\beta_i<\aleph_1$ with the purpose that X'

will be $\{x_i \mid i<\aleph_1\}$ and C will be $\{\beta_i \mid i<\aleph_1\}$. Suppose x_j, β_j have been defined for every $j<i$. If i is a limit ordinal let $\beta_i=\sup(\{\beta_j \mid j<i\})$, otherwise let β_i be an ordinal greater than any ordinal in the set $\{\gamma \mid y_\gamma \in \bigcup\{F_k(x_j) \mid k, j<i$ and $x_j \in D(F_k)\}\} \cup \{\beta_j \mid j<i\}$. Let $x_i \in X-\{x_j \mid j<i\}-\bigcup\{F_k^{-1}(y_{\beta_j}) \mid k<i,\ j\leq i$ and $y_{\beta_j} \in R(F_k)\}$.

Let $C=\{\beta_i \mid i<\aleph_1\}$ and $X'=\{x_i \mid i<\aleph_1\}$. Clearly X' is uncountable and C is a club. Let $Y'=\{y_\beta \mid \beta \in C\}$. Suppose by contradiction there is an uncountable 1-1 $h \subseteq X' \times Y'$ and $l \in \{0, 1\}$ such that for every $x_1, x_2 \in \mathrm{Dom}(h)$: if $f(x_1, x_2)=1$, then $g(h(x_1), h(x_2))=l$. Let $F \subseteq X \times Y$ be the closure of h in $X \times Y$. Since $f^{-1}(1)$ is open and $g^{-1}(l)$ is closed, F satisfies (*), hence for some i, $F=F_i$. Let $\langle x_j, y_{\beta_k}\rangle \in h$ be such that $i<j$, β_k, $x_j \in D(F)$ and $y_{\beta_k} \in R(F)$. If $j \geq k$, then we picked x_j not in $F^{-1}(y_{\beta_k})$, a contradiction. If $j<k$ and k is a successor, then we picked β_k such that $y_{\beta_k} \notin F_i(x_j)$, a contradiction. Suppose $j<k$ and k is a limit ordinal. Let $j<k_1<k$ be a successor ordinal. Then $\beta_k>\beta_{k_1}$ and β_{k_1} is greater than any element of the set $\{\gamma \mid y_\gamma \in F_i(x_j)\}$, hence $\beta_k \notin \{\gamma \mid y_\gamma \in F_i(x_j)\}$, a contradiction. This concludes the proof of the lemma. □

Proof of Lemma 4.5. Lemma 4.5 is very similar to the first part in the proof of Lemma 3.3. We leave it to the reader to translate the first part of the proof of 3.3 to a proof of this lemma.

We next have to deal with the following case. Suppose P_α, C_α have been defined, and the αth task is as follows: we are given a P_α-name of a c.c.c. forcing set R_α of power $\leq\aleph_1$, and we have either to add a generic filter to R_α or to destroy the c.c.c.-ness of R_α. If $\Vdash_{R_\alpha}$ "Y_α does not contain uncountable homogeneous sets", then $P_{\alpha+1}=P_\alpha * R_\alpha$ and $C_{\alpha+1}=C_\alpha$. We deal with the case when there is $r \in R_\alpha$ such that $r \Vdash_{R_\alpha}$ "Y_α contains an uncountable homogeneous set". In this case we will construct a c.c.c. forcing set Q_α such that $\Vdash_{Q_\alpha}$ "R_α is not c.c.c., and Y_α does not contain an uncountable homogeneous set".

Lemma 4.6. *Let $\langle Y, g\rangle$ be an OC pair that does not contain uncountable homogeneous subsets, let R be a c.c.c. forcing set and $r \in R$ be such that $r \Vdash_R$ "Y contains an uncountable homogeneous set". Then there is a c.c.c. forcing set Q of power $\aleph_1$ such that $\Vdash_Q$ "R is not c.c.c. and Y does not contain uncountable homogeneous subsets".*

Remark. The proof resembles Theorem 2.4 in [1].

Proof. Let M be a model with universe $\aleph_1$ encompassing the space Y, the function g and enough set theory. Let $C_M=\{\alpha \mid M \restriction \alpha \prec M\}$. Let $\tilde{B}$ be an R-name, $r \in R$ and $l \in \{0, 1\}$ be such that $r \Vdash_R$ "$\tilde{B}$ is an uncountable l-colored subset of Y". W.l.o.g. $r=0_{\mathbb{R}}$ and $l=1$. We choose a sequence $\{\langle r_\alpha, y_\alpha^1, y_\alpha^2\rangle \mid \alpha<\aleph_1\}$ such that: for every α, $r_\alpha \Vdash_R$ "$y_\alpha^1, y_\alpha^2 \in \tilde{B}$"; for every $\alpha<\beta<\aleph_1$, $y_\alpha^1<y_\alpha^2<y_\beta^1$, and

$\{y_\alpha^1, y_\alpha^2, y_\beta^1, y_\beta^2\}$ is C_M-separated. W.l.o.g. there are basic open sets U_1, U_2 of Y such that for every $\alpha<\aleph_1$ and $i\in\{1,2\}$, $y_\alpha^i\in U_i$ and $g(U_1\times U_2)=\{1\}$.

We define Q as follows. $Q=\{\sigma\in P_{\aleph_0}(\aleph_1)\mid$ for every distinct $\alpha,\beta\in\sigma$ there is $i\in\{1,2\}$ such that $g(y_\alpha^i, y_\beta^i)=0\}$. Note that the last clause in the definition of Q is just the 'explicit contradiction' clause, hence if $\alpha,\beta\in\sigma\in Q$ are distinct, then r_α and r_β are incompatible in R. The partial ordering in Q is of course: $\sigma\leqslant\tau$ if $\sigma\subseteq\tau$.

The proof that Q is c.c.c. resembles the analogous proof in Section 3. The argument why $\Vdash_Q$ "R is not c.c.c." is also as in Section 3.

Let us show that $\Vdash_Q$ "Y does not contain uncountable homogeneous subsets". Suppose by contradiction that the above is not true. Then there is a sequence $\{\langle q_\beta, y_\beta\rangle \mid \beta<\aleph_1\}$ and $l\in\{0,1\}$ such that for every $\alpha<\beta<\aleph_1$, $q_\alpha\in Q$, $y_\alpha\neq y_\beta$ and if $q_\alpha\cup q_\beta\in Q$, then $g(y_\alpha, y_\beta)=l$. As usual we uniformize the sequence $\{\langle q_\beta, y_\beta\rangle \mid \beta<\aleph_1\}$ as much as possible, hence we may assume that $q_\beta=\{\alpha_0,\ldots,\alpha_{k-1},\alpha_k^\beta,\ldots,\alpha_{k-1}^\beta\}$ where for every $\beta<\gamma$, $\alpha_0<\cdots<\alpha_{k-1}<\alpha_k^\beta<\cdots<\alpha_{n-1}^\beta<\alpha_k^\gamma$. Let us denote $y^i_{\alpha_j^\beta}$ by $y(\beta,j,i)$, and $y^i_{\alpha_j}$ by $y(j,i)$. Let $\mathbf{y}=\langle y(0,0), y(0,1),\ldots, y(k-1,1)\rangle$, $\mathbf{y}(\beta,j)=\langle y(\beta,j,0), y(\beta,k,1)\rangle$, $\mathbf{y}(\beta)=\mathbf{y}^\frown\mathbf{y}(\beta,k)^\frown\cdots{}^\frown\mathbf{y}(\beta,n-1)$ and $\mathbf{z}(\beta)=\mathbf{y}(\beta)^\frown\langle y_\beta\rangle$. Recall that if we take two pairs $\mathbf{y}(\beta,i)$ and $\mathbf{y}(\beta,j)$ where $i\neq j$, then either their first or their second coordinates have color 0, i.e. either $g(y(\beta,i,0), y(\beta,j,0))=0$ or $g(y(\beta,i,1), y(\beta,j,1))=0$. Hence by more uniformization we can assume that there are m_{ij}'s for $k\leqslant i<j\leqslant n-1$ such that for every β, i and j, $g(y(\beta,i,m_{ij}), y(\beta,j,m_{ij}))=0$, and that there are basic sets U_i^m, $i=k,\ldots,n-1$, $m=0,1$, such that for every β, $y(\beta,i,m)\in U_i^m$ and for every $k\leqslant i<j\leqslant n-1$, $g(U_i^{m_{ij}}\times U_j^{m_{ij}})=0$. Let F be the closure of $\{\mathbf{z}(\beta)\mid\beta<\aleph_1\}$ in Y^{n+1}, let $\gamma_0\in C_M$ be such that F is definable in M from a parameter belonging to γ_0. Let β be such that all the elements of $\mathbf{z}(\beta)$ except the first k of them do not belong to γ_0. We duplicate $\mathbf{z}(\beta)$. Note that $\mathbf{y}(\beta)$ is separated. Let E be the C_m-slice to which y_β belongs. Hence there is at most one element of $\mathbf{y}(\beta)$ which belongs to E. To simplify the notation let us assume that this element is $y(\beta,k,0)$. Hence by the duplication argument, and since we know that Y does not contain homogeneous uncountable subsets, we can find $\mathbf{z}^i=\mathbf{y}^\frown\langle y^i(k,0), y^i(k,1),\ldots, y^i(n-1,0), y^i(n-1,1), y^i\rangle\in F$, $i=1,2$, such that $g(y^1,y^2)\neq l$, and $g(y^1(t,j), y^2(t,j))=0$ for $\langle t,j\rangle=\langle k,1\rangle$, $\langle k+1,0\rangle$, $\langle k+1,1\rangle,\ldots,\langle n-1,0\rangle,\langle n-1,1\rangle$. Since g is continuous we can find neighbourhoods V_1, V_2 of $\mathbf{z}^1$, $\mathbf{z}^2$ respectively such that the same equalities hold whenever we pick $\mathbf{z}_1\in V_1$ and $\mathbf{z}_2\in V_2$. Let $\mathbf{z}(\alpha)\in V_1$ and $\mathbf{z}(\beta)\in V_2$ it is easy to check that $q_\alpha\cup q_\beta\in Q$ but $g(y_\alpha, y_\gamma)\neq l$, a contradiction. □

This concludes the proof of Theorem 4.1.

The tail method will be used again in Section 9 and 10. The reader can check by himself that combining the club method, the explicit contradiction method and the tail method one can e.g., get the following consistency result that was mentioned in Section 2.

Theorem 4.7. MA + SOCA + $\exists A$ (*A is increasing and rigid*) *is consistent.*

5. Enlarging the continuum beyond $\aleph_2$

According to our presentation in the previous sections, we always had to assume CH in the ground model in order to apply the club method. Thus in the resulting models of set theory $2^{\aleph_0}$ had to be equal to $\aleph_2$.

The goal of this section is to find a weaker assumption under which the club method can work. Hence we will be able to prove that some of the axioms considered in the previous sections are consistent with $2^{\aleph_0} > \aleph_2$.

Indeed CH was used in more than one way. In Theorem 1.5 we used CH in order to prove the following claim. If $\{F_i \mid i \in I\}$ is a family of closed subsets of X, and X is not the union of countably many F_i's, then there is an uncountable $X' \subseteq X$ which intersects every F_i in at most countably many points. In Section 3 we used CH in order to preassign colors, and in Lemma 4.4 we used CH in still another way.

It so happens that MA + OCA implies $2^{\aleph_0} = \aleph_2$. However, MA + SOCA + $2^{\aleph_0} > \aleph_2$ is consistent. We present the new method by means of an example. We will show that MA + SOCA + $(2^{\aleph_0} > \aleph_2)$ is consistent. The reader will be able to check that the consistency of MA + NWD2 + $(2^{\aleph_0} > \aleph_2)$ can be proved by the same method. The proof that MA + OCA $\Rightarrow 2^{\aleph_0} = \aleph_2$ will be presented in Section 11. But some questions remain open, and we will mention them in Section 11.

In view of this section and Section 11, certainly MA + SOCA $\not\Rightarrow$ OCA, hence we do not have to prove Lemma 4.4 in the absence of CH, however since the proof exemplifies what can be done without CH we take the liberty to present its short proof. This is done in Lemma 5.5.

Let $A \subseteq B$ denote that $|A - B| \leq \aleph_0$. Let M be a model in a countable language such that $|M| \supseteq \aleph_1$, and let D be a finite subset of $|M|$; we denote $C_{M,D} \stackrel{\text{def}}{=} \{\alpha \in \aleph_1 \mid \text{there exists } N < M \text{ such that } D \subseteq |N| \text{ and } \alpha = |N| \cap \aleph_1\}$. Clearly $C_{M,D}$ contains a club. A club C of $\aleph_1$ is called M-thin, if for every finite $D \subseteq |M|$, $C \subseteq C_{M,D}$.

Let us reconstruct the proof of Theorem 1.1. The central point in the proof was to construct for a given SOC f of a second countable space X, a forcing set $P_{X,f}$ which adds to X an uncountable 1-colored subset. To do this we constructed a model M which included all the relevant information about X and f, and defined $P_{X,f}$ to be the set of all finite, C_M-separated, 1-colored subsets of X. In the proof that $P_{X,f}$ was c.c.c., the only property of C_M that was used, was its M-thinness.

Let W be a universe in which $2^{\aleph_1} > \aleph_2$. In order to be able to repeat the construction of Theorem 1.1. starting with W as the ground model, we thus need that W will have the following property. If $P \in W$ is a c.c.c. forcing set of power $< 2^{\aleph_1}$, $G \subseteq P$ is a generic filter and $M \in W[G]$ is a model which is constructed for some $\langle X, f\rangle \in W[G]$, then $W[G]$ contains an M-thin club.

We will show that such W's can be constructed, and in fact, the W's that we construct will contain M-thin clubs for a wider set of M's rather than for just those M's that come from some $\langle X, f\rangle$. This fact will be important in other applications of the method.

We define the countable closure of a set A. For $i \leq \aleph_1$ we define by induction $A^{(i)}: A^{(0)} = A$; if δ is a limit ordinal, then $A^{(\delta)} = \bigcup_{i<\delta} A^{(i)}$; and $A^{(i+1)} = A^{(i)} \cup \{B \subseteq A^{(i)} \mid |B| \leq \aleph_0\}$. Let $A^c = A^{(\aleph_1)}$, A^c is called the countable closure of A. For a model M, let M^c be the following model. $|M^c| = |M|^c$, the relations in M^c are those of M, and in addition: the belonging relation on $|M|^c$, and a unary predicate which represents $|M|$ in M^c. A model of the form M^c, where $\|M\| < 2^{\aleph_1}$ and M has a countable language is called a low model.

Axiom A1. If P is a c.c.c. forcing set of power $<2^{\aleph_1}$, and if M is a P-name of a model, such that $\Vdash_P$ "M is a low model", then $\Vdash_P$ "there is an M-thin club".

Proposition 5.1. *Let W be a universe of set theory which satisfies* A1, *then there is a c.c.c. forcing set Q of power* $2^{\aleph_1}$ *such that* $W^Q \models$SOCA.

To prove the above proposition one has to reexamine the proof of Theorem 1.1. and check the following fact. Let f be a SOC of X, and suppose X does not contain uncountable 0-colored sets. W.l.o.g. $X = \aleph_1$; let $M = \langle \aleph_1 \cup \mathcal{B}; \in, <, f\rangle$ where $\mathcal{B}$ is a countable base for X, $\in$ is the belonging relation between elements of X and elements of $\mathcal{B}$, and $<$ is the ordering relation on $\aleph_1$; and let C be an M^c-thin club. Then if P is the set of all finite C-separated 1-colored subsets of X, then P is c.c.c., and $\Vdash_P$ "X contains an uncountable 1-colored subset". We leave all the other details to the reader.

Our next goal is to construct a W in which $2^{\aleph_1} > \aleph_2$ and which satisfies A1. Let us explain how such a W is constructed. We start with a universe V which satisfies CH. Let λ be a regular cardinal in V such that $\lambda^{\aleph_1} = \lambda$.

We define a countable support iteration $\{\langle P_\alpha, \pi_\alpha\rangle \mid \alpha \leq \lambda\}$ in which each π_α is the name of a forcing set which adds a club C to $\aleph_1$, such that C is almost contained in every club which belongs to V^{P_α}. We will show that in V^{P_λ}, $2^{\aleph_1} = \lambda$ and A1 holds.

Let $P_{Cb} = \{\langle D, f\rangle \mid D$ is a closed and bounded subset of $\aleph_1$, F is a club in $\aleph_1$ and $D \subseteq F\}$. let $\langle D_1, F_1\rangle$, $\langle D_2, F_2\rangle \in P_{Cb}$, then $\langle D_1, F_1\rangle \leq \langle D_2, F_2\rangle$ if D_1 is an initial segment of D_2 and $F_2 \subseteq F_1$.

Proposition 5.2 (R. Jensen). (a) P_{Cb} *is* ω*-closed.*

(b) (CH) P_{Cb} *is* $\aleph_2$*-c.c.*

(c) (CH) *Let* $\langle\{P_{Cb}(\alpha) \mid \alpha \leq \lambda\}, \{\pi_{Cb}(\alpha) \mid \alpha < \lambda\}\rangle$ *be a countable support iteration*; $P_{Cb}(0)$ *is a trivial forcing set, and for every* $\alpha < \lambda$, $\pi_{Cb}(\alpha)$ *is a* $P_{Cb}(\alpha)$*-name of the forcing set* P_{Cb} *of the universe* $V^{P_{Cb}(\alpha)}$. *Then* $P_{Cb}(\lambda)$ *is* ω*-closed and* $\aleph_s$*-c.c., and if* $\lambda^{\aleph_1} = \lambda$, *then* $\Vdash_{P_{Cb}(\lambda)} 2^{\aleph_1} = \lambda$.

Proof. Well known.

Lemma 5.3. *Let $P, Q \in V$ be forcing sets such that P is c.c.c. in V and Q is ω-closed in V. Then V^P is closed under ω-sequences in $V^{P\times Q}$.*

Proof. It suffices to show that every ω-sequence of ordinals in $V^{P\times Q}$ belongs to V^P. Let τ be a $P\times Q$-name of an ω-sequence of ordinals. We show that for every $q_0 \in Q$ there is $q_1 \geq q_0$ with the following property:

(*) For every $p_0 \in P$ and $n \in \omega$ there is $p_1 \geq p$ and an ordinal α such that $\langle p_1, q_1\rangle \Vdash \tau(n) = \alpha$. Suppose by contradiction $q^0 \in Q$ and there is no $q_1 \geq q^0$ which satisfies (*). We define by induction on i a sequence $\{\langle q_i, p_i, n_i\rangle \mid i < \aleph_1\}$. Let $q_0 \geq q^0$, $p_0 \in P$, $n_0 \in \omega$ be such that there is α_0 such that $\langle p_0, q_0\rangle \Vdash \tau(n_0) = \alpha_0$. Suppose $\langle q_j, p_j, n_j\rangle$ has been defined for every $j < i$. Let $q^i \geq q_j$ for every $j < i$. Since $q^1 > q^0$, (*) does not hold for q^i, and hence there is $p^i \in P$ and $n_i \in \omega$ such that for every $p \geq p^i$ and an ordinal α, $\langle p, q^i\rangle \nVdash \tau(n_i) = \alpha$. Let $\langle p_i, q_i\rangle \geq \langle p^i, q^i\rangle$ and α_i be an ordinal such that $\langle p_i, q_i\rangle \Vdash \tau(n_i) = \alpha_i$. Let $i < j$ be such that $n_i = n_j$; we show that p_i and p_j are incompatible. Suppose by contradiction $r \geq p_i, p_j$. Hence $\langle r, q_i\rangle \Vdash \tau(n_i) = \alpha_i$. But $q_i \leq q^j$, hence $\langle r, q^j\rangle \Vdash \tau(n_i) = \alpha_i$. But $r \geq p_j \geq p^j$, hence there is $p \geq p^j$ and α such that $\langle p, q^j\rangle \Vdash \tau(n_j) = \alpha$. This contradicts the choice of p^j, q^j and n_j. Let n be such that $|\{i \mid n_i = n\}| = \aleph_1$, hence $\{p_i \mid n_i = n\}$ is an uncountable antichain in P, a contradiction.

For a $P\times Q$-name τ of an ω-sequence of ordinals let $D_\tau = \{q \in Q \mid q$ satisfies (*)$\}$. We have thus shown that for every τ as above D_τ is dense in Q.

Let $G \subseteq P\times Q$ be a generic filter, and let $a \in V[G]$ be an ω-sequence of ordinals. Let G_1, G_2 be the restrictions of G to P and Q respectively. We show that $a \in V[G_1]$. Let τ be a $P\times Q$-name of a. Let $q \in D_\tau \cap G_2$. For every $n \in \omega$, let $p_n \in G_1$ and α_n be such that $\langle p_n, q\rangle \Vdash \tau(n) = \alpha_n$. Hence $a = \langle \alpha_n \mid n \in \omega\rangle$ and clearly $a \in V[G_1]$. $\square$

Let $\langle\{P_\alpha \mid \alpha \leq \lambda\}, \{\pi_\alpha \mid \alpha < \lambda\}\rangle$ be an iteration of length λ. We denote by $\hat{P}_\beta$ and P_β-name of the iteration which is formed from the sequence of names $\{\pi_\alpha \mid \beta \leq \alpha < \lambda\}$. Hence $P_\beta * \hat{P}_\beta \cong P$.

Lemma 5.4. *Let $V \models \mathrm{CH}$ and let λ be a regular cardinal in V such that $\lambda^{\aleph_1} = \lambda$. Then $\Vdash_{P_{\mathrm{Cb}}(\lambda)} \mathrm{A1}$.*

Proof. Let $Q_0 = P_{\mathrm{Cb}}(\lambda)$. Let $G \subseteq Q_0$ be a generic filter and $W = V[G]$. Let $P \in W$ be a c.c.c. forcing set of power $< 2^{\aleph_1}$, let $H \subseteq P$ be generic and $U = W[H]$. Let $M \in U$ be a model in a countable language such that $\aleph_1 \subseteq |M|$ and $\|M\| < 2^{\aleph_1}$. Since Q_0 is $\aleph_2$-c.c., $|P| < 2^{\aleph_0}$ and $(2^{\aleph_1})^{(W)} = \lambda$ it follows that for some $\alpha < \lambda$, $P \in V[G \cap P_{\mathrm{Cb}}(\alpha)]$. Similarly for some $\alpha \leq \beta < \lambda$, $M \in V[G \cap P_{\mathrm{Cb}}(\beta)][H]$. Let $G_1 = G \cap P_{\mathrm{Cb}}(\beta)$, $V_1 = V(G_1)$ and $Q_1 = V_{G_1}(\hat{P}_{\mathrm{Cb}}(\beta))$, and let G_2 be the generic filter of Q_1 determined by G. Hence P is c.c.c. in V_1, Q_1 is ω-closed in V_1, $H \times G_2$ is $P\times Q_1$-generic and $V_1[H][G_2] = U$. By the previous lemma $V_1[H]$ is closed under ω-sequences in U, and thus since $M \in V_1[H]$, also $M^c \in V_1[H]$. Let D be the club of $\aleph_1$ which is added by the restriction of G to $\pi_{\mathrm{Cb}}(\beta)$, hence D is almost contained in every club of V_1. However since H is c.c.c. in V_1, every club of $V_1[H]$ contains a club of V_1, and hence D is almost contained in every club of V_1, and hence D is almost contained in every club of $V_1[H]$, and obviously this implies that D is M^c-thin. □

Lemma 5.5. *Lemma* 4.4 *is true in a universe* V^P *where* $V \vDash \mathrm{A1}$ *and* P *is a c.c.c. forcing set of power* $<2^{\aleph_1}$.

Proof. Let $\langle X, f\rangle$, $\langle Y, g\rangle$, $\{y_\beta \mid \beta<\aleph_1\}$ be as in 4.4, and let M be a model which encodes X, f, Y, g and $\{y_\beta \mid \beta<\aleph_1\}$. Let C be M^c-thin, and let $\{E_\alpha \mid \alpha<\aleph_1\}$ be an enumeration of the C-slices in an increasing order. Let D be a club in $\aleph_1$ such that $\{\alpha \mid E_\alpha \cap \{y_\gamma \mid \gamma \in D\} = \emptyset\}$ is uncountable, and let $X' \subseteq X$ be an uncountable set such that for every $\alpha<\aleph_1$ if $X' \cap E_\alpha \neq \emptyset$, then $\{y_\gamma \mid \gamma \in D\} \cap E_\alpha = \emptyset$. We show that D and X' are as required. Let $Y' = \{y_\gamma \mid \gamma \in D\}$, and suppose by contradiction for some $h \subseteq X' \times Y'$ and $l \in \{0, 1\}$, h is uncountable and 1-1 and whenever $x_1, x_2 \in X'$ and $f(x_1, x_2) = 1$, then $g(h(x_1, h(x_2)) = l$. Let $F = \mathrm{cl}(h)$. Then F is definable in M^c. Using the notation of 4.4, F satisfies (*). Hence by the proof of 4.4 for all but countably many elements $x \in X'$, $F(x)$ lies in the same C-slice that x does. This contradicts the choice of X' and Y'. □

6. MA, OCA and the embeddability relation on $\aleph_1$-dense real order types

Let $A \subseteq \mathbb{R}$. A is $\aleph_1$-dense if it has no first and no last element, and if between any two members of A there are exactly $\aleph_1$ members of A. If $A, B \subseteq \mathbb{R}$, let $A^* = \{-a \mid a \in A\}$, let $A \cong B$ mean that the structures $\langle A, <\rangle$ and $\langle B, <\rangle$ are isomorphic, let $A \preccurlyeq B$ mean that $\langle A, <\rangle$ is embeddable in $\langle B, <\rangle$, and let $A \perp B$ mean that for no $\aleph_1$-dense $C \subseteq \mathbb{R}: C \preccurlyeq A$ and $C \preccurlyeq B$. $f: A \to B$ is *order preserving* (OP), if for every $a_1, a_2 \in A$, $a_1 < a_2 \Rightarrow f(a_1) < f(a_2)$; it is *order reversing* (OR), if for every $a_1, a_2 \in A$, $a_1 < a_2 \Rightarrow f(a_2) < f(a_1)$; f is *monotonic* if F is either OP or OR. Let $K = \{A \subseteq \mathbb{R} \mid A \text{ is } \aleph_1\text{-dense}\}$. $A \subseteq \mathbb{R}$ is *homogeneous*, if for every $a, b \in A$ there is an automorphism f of $\langle A, <\rangle$ such that $f(a) = b$. Let $K^H = \{A \in K \mid A$ is homogeneous$\}$. It follows easily from ZFC that for every $A \in K$ there is $B \in K^H$ such that $A \subseteq B$. Let $\{A_i \mid i<\alpha\} \subseteq K$, and let $B \in K$. We say that B is a *shuffle* of $\{A_i \mid i<\alpha\}$ if there are A_i^1 such that $A_i^1 \cong A_i$, $B = \bigcup_{i<\alpha} A_i^1$ and for every $i<\alpha$ and $b_1, b_2 \in B$ such that $b_1 < b_2$ there is a $a \in A_i^1$ such that $b_1 < a < b_2$.

Let $A \in K$; $B \in K$ is a *mixing* of A if for every rational interval I there is A_I such that $A_I \subseteq I$, $A_I \cong A$ and $B = \bigcup \{A_I \mid I$ is a rational interval$\}$.

Baumgartner [2] proved that it is consistent that all members of K are isomorphic. Shelah [1] invented the club method and used it to show that $\mathrm{MA} + \aleph_1 < 2^{\aleph_0}$ does not imply Baumgartner's axiom (BA). He constructed a universe in which MA holds but $\mathbb{R}$ contains an entangled set. An entangled set A

has the property that there is no uncountable monotonic function g with no fixed points such that $\mathrm{Dom}(g)$, $\mathrm{Rng}(g) \subseteq A$. Thus A is rigid in a very strong sense. The consistency of $\mathrm{MA}+\aleph_1<2^{\aleph_0}$ with the existence of an increasing set was proved by Avraham in [1]. An increasing set is an analogue of an entangled set when monotonic functions are replaced by OR functions.

It was natural to ask how much freedom do we have in determining the structure of the category whose members are the elements of K and whose morphisms are the monotonic functions. In this section we start investigating such questions under the assumption of $\mathrm{MA}+\aleph_1<2^{\aleph_0}$.

We shall first see that $\mathrm{MA}_{\aleph_1}$ already implies many properties of K. Next we shall see that $\mathrm{MA}_{\aleph_1}+\mathrm{OCA}$ determines K quite completely, namely if we conjunct $\mathrm{MA}_{\aleph_1}+\mathrm{OCA}$ with the existence of an increasing set, then K^H consist up to isomorphism of three elements and every element of K is built from these elements in a simple way. On the other hand $\mathrm{MA}_{\aleph_1}+\mathrm{OCA}+\neg\mathrm{ISA}$ implies Baumgartner's axiom.

Theorem 6.1. ($\mathrm{MA}_{\aleph_1}$). (a) *If $A \in K^H$, $a_1, b_1, a_2, b_2 \in A$, and $a_1<b_1$ and $a_2<b_2$, then there is an automorphism f of $\langle A, <\rangle$ such that $f(a_1)=a_2$ and $f(b_1)=b_2$.*

(b) *Let $A, B \in K$ and let $\{g_i \mid i \in \omega\}$ be a family of OP functions such that for every $a \in A$ and $b_1<b_2$ in B there is $i \in \omega$ such that $g_i \in B \cap (b_1, b_2)$. Then $A \preccurlyeq B$.*

(c) *Let A, B and $\{g_i \mid i \in \omega\}$ be as in* (b), *and suppose in addition that for every $b \in B$ and $a_1<a_2$ in A there is $i \in \omega$ such that $g_i^{-1}(b) \in A \cap (a_1, a_2)$. Then $A \cong B$.*

(d) *If $A, B \in K^H$, $A \preccurlyeq B$ and $B \preccurlyeq A$, then $A \cong B$. (Hence $\preccurlyeq$ is a partial ordering of $K^H/\cong$.)*

(e) *If $A \in K^H$, then A is isomorphic to every non-empty open interval of A.*

(f) *Let $\{A_i \mid i<\alpha \leqslant \omega\} \subseteq K^H$. Then* (a) *all shuffles of $\{A_i \mid i<\alpha\}$ are isomorphic and they belong to K^H. In particular,if all the A_i's are isomorphic to some fixed A, then every shuffle of $\{A_i \mid i<\alpha\}$ is isomorphic to A; and* (b) *if B is a shuffle of $\{A_i \mid i<\alpha\}$ and $C \in K^H$ and for every $i<\alpha$, $A_i \preccurlyeq C$, then $B \preccurlyeq C$.*

(g) *If $A \in K$ and B_1, B_2 are mixings of A, then $B_1 \cong B_2$ and $B_1 \in K^H$, and if $C \in K^H$ and $A \preccurlyeq C$, then $B_1 \preccurlyeq C$.*

(h) *If $A \in K$ and for every $B \in K$ $A \preccurlyeq B$, then $A \in K^H$.*

(i) *If for every $A, B \in K$ $A \preccurlyeq B$, then* BA *holds.*

(j) *If $|K^H/\cong|=1$, then* BA *holds.*

Proof. All parts of 6.1 follow easily from (b) and (c). We prove (c), (a) and (j), and leave the other parts to the reader.

(c) Let $P=\{f \in P_{\aleph_0}(A \times B) \mid f$ is OP, and for every $a \in \mathrm{Dom}(f)$ there is g_i such that $f(a)=g_i(a)\}$. $f \leqslant g$ if $f \subseteq g$. It is easy to see that P is c.c.c.. It is also easy to see that for every $a \in A$, $D_a \stackrel{\text{def}}{=} \{f \in P \mid a \in \mathrm{Dom}(f)\}$ is dense in P, and for every $b \in B$, $D^b \stackrel{\text{def}}{=} \{f \in P \mid b \in \mathrm{Rng}(f)\}$ is dense in P, hence if G is a filter in P which intersects all D_a's and D^b's, then $\bigcup\{f \mid f \in G\}$ is an isomorphism between A and B.

(a) It suffices to show that for every non-empty open intervals I_0 and J_0 of A, $I_0 \cong J_0$. Let I_0, J_0 be such intervals. For every interval $J \subseteq J_0$ let $\{\langle I_i^J, g_i^J\rangle \mid i \in \omega\}$ be such that I_i^J is an interval of A, g_i^J is an automorphism of A, $g_i^J(I_i^J) \subseteq J$, and $\bigcup_{i\in\omega} I_i^J = A$. Let $\mathcal{J}$ be a countable base for the order topology of J consisting of open intervals and let $\mathcal{G}_0 = \{g_i^J \mid J \in \mathcal{J}, i \in \omega\}$. Then I_0, J_0 and $\mathcal{G}_0$ satisfy the conditions of part (b). By a symmetric argument there is a family $\mathcal{G}_1$ such that J_0, I_0 and $\mathcal{G}_1$ satisfy the conditions of part (b). Hence I_0, J_0 and $\mathcal{G}_0 \cup \{g^{-1} \mid g \in \mathcal{G}_1\}$ satisfy the conditions of part (c), hence $I_0 \cong J_0$.

(j) For every $A \in K$ let A^m be some canonical mixing of A. Let $A, B \in K$, and we show that $A \preccurlyeq B$. Let B_1 be a nonempty open interval of B. $A \preccurlyeq A^m \cong B_1^m$, hence let $g_{B_1} : A \to B_1^m$ be OP. Let $B_1^m = \bigcup_{i\in\omega} B^i$ where $B^i \cong B_1$, and let $g_i : B_1 \to B^i$ be an isomorphism. Let $h_{B_1,i} = g_i^{-1} \circ g_{B_1}$, hence $h_{B_1,i} \subseteq A \times B_1$. Let $\{B_j \mid j \in \omega\}$ be a dense family of non-empty open intervals of B. Clearly $A, B, \{h_{B_j,i} \mid j, i \in \omega\}$ satisfy the conditions of part (b), hence by (b), $A \preccurlyeq B$.

It follows from part (i) that BA holds. □

Theorem 6.2. *Let* $V \models \mathrm{MA} + \mathrm{OCA} + \mathrm{ISA}$, *and let* $A \in K$ *be an increasing set dense in* $\mathbb{R}$. *Then:*

(a) $A, A^*, A \cup A^* \in K^H$, *and every member of* K^H *is isomorphic to one of these sets.* (*Of course* $A \perp A^*$.)

(b) *If* $K \ni B \subseteq A$, *then* $B \cong A$.

(c) *If* $A \cup A^* \subseteq B \in K$, *then* $B \cong A \cup A^*$. (*Hence* $A \cup A^*$ *is universal in* K.)

(d) *If* $B \in K$ *is dense in* $\mathbb{R}$, *then there is a nwd set* $C \subseteq \mathbb{R}$, *a countable ordered set* $\langle L, <\rangle$, *and for every* $l \in L$ *a member* $A_l \in K^H$ *such that* $B - C \cong \sum_{l\in L} A_l$, *where* $\sum_{l\in L}$ *denotes the ordered sum of linearly ordered sets.*

(e) *If* $B \in K$, *then either* $B \cong A + 1 + A^*$ *or* $B \cong A^* + 1 + A$, *or* B *can be represented in the form* $B_1 \cup B_2$ *where* $B_1 \cap B_2 = \emptyset$, $B_1 \cong A$ *or* $B_1 = \emptyset$, *and* $B_2 \cong A^*$ *or* $B_2 = \emptyset$.

Proof. The proofs of all parts are easy, as an example we prove (b). Let $K \ni B \subseteq A$. Let I, J be non-empty open intervals of B and A respectively, and let $f^{I,J} : I \to J$ be a 1-1 onto function. By OCA, $f^{I,J} = \bigcup \{f_i^{I,J} \mid i \in \omega\}$, where for every i, $f_i^{I,J}$ is monotonic. Since $B \subseteq A$ and A is an increasing set, each $f_i^{I,J}$ is OP. Let $\mathcal{I}$ be a countable dense set of non-empty open intervals of B, i.e. for every non-empty open interval I of B there is $I_1 \in \mathcal{I}$ such that $I_1 \subseteq I$. Similarly let $\mathcal{J}$ be a countable dense set of non-empty open intervals of A, and let $\mathcal{G} = \{f_i^{I,J} \mid I \in \mathcal{I}, J \in \mathcal{J}$ and $i \in \omega\}$. Clearly B, A and G satisfy the conditions of 6.1(c) and hence $B \cong A$. □

Question 6.3. Construct a model of ZFC in which for every $A, B \in K$, $A \preccurlyeq B$, but in which BA does not hold.

Let NA denote the following axiom: $(\forall A, B \in K)((\exists C \in K)(C \preccurlyeq A \wedge C \preccurlyeq B)$. Clearly $\mathrm{MA} + \mathrm{OCA} \Rightarrow (\mathrm{NA} \Leftrightarrow \neg\mathrm{ISA})$. In Section 7 we shall see that MA + OCA +

NA is consistent. We conclude this section by showing that MA+OCA+NA⇒ BA.

Proposition 6.4. MA+OCA+NA⇒BA.

Proof. Let $A, B \in K$ and we show that $A \preccurlyeq B$. Let B_1 be a non-empty open interval of B. By NA there are $C_1, C_2 \subseteq B_1$ such that $C_1 \in K$ and $C_2 \cong C_1^*$. Let $f: A \to C_1$ be a 1-1 function. Then f can be represented as a contable union of monotonic functions $f = \bigcup_{i \in \omega} f_i$. Let $g: C_1 \to C_2$ be an OR onto function; for every $i \in \omega$ let $h_{B_1,i} = f_i$ if f_i is OP, and $h_{B_1,i} = g \circ f_i$ if f_i is OR. Let $\{B_j \mid j \in \omega\}$ be a dense family of intervals of B. Then A, B, $\{h_{B_j,i} \mid j, i \in \omega\}$ satisfy the conditions of 6.1(b). Hence $A \preccurlyeq B$. □

7. Relationship with the weak continuum hypothesis

Our purpose in this section is to present a forcing set which makes two $\aleph_1$-dense sets of real numbers near, that is, given $A, B \in K$ we want to add $C \in K$ such that $C \preccurlyeq A$ and $C \preccurlyeq B$. Let $N(A, B) \equiv (\exists C \in K)\,(C \preccurlyeq A \wedge C \preccurlyeq B)$, and let the nearness axiom be as follows.

Axiom NA. $(\forall A, B \in K)\, N(A, B)$.

Obviously BA⇒NA. One can ask whether these two axioms are equivalent. We will show in this section that BA⇒ $2^{\aleph_0} = 2^{\aleph_1}$ whereas CON(NA+ $2^{\aleph_0} < 2^{\aleph_1}$) holds, so NA⇏BA.

Let WCH denote the axiom that $2^{\aleph_0} < 2^{\aleph_1}$. $A \in K$ is *prime* if for every $B \in K: A \preccurlyeq B$; it is *universal* if for every $B \in K: B \preccurlyeq A$. $\{A_i \mid i \in I\} \subseteq K$ is a *prime family*, if for every $B \in K$ there is $i \in I$ such that $A_i \preccurlyeq B$.

Theorem 7.1 (WCH). (a) *K does not contain a prime element. Moreover there is no prime family of power $<2^{\aleph_1}$.*

(b) *For every $A \in K$ there is $B \subseteq A$ such that $B \in K$ and $A \not\preccurlyeq B$.*

Proof. Let $T = {}^{\aleph_1 >}2$ be the tree of binary sequences of length $<\aleph_1$. Clearly $|T| = 2^{\aleph_0}$. Let $\{a_\nu \mid \nu \in T\}$ be a 1-1 function from T to $\mathbb{R}$. For every $\eta \in {}^{\aleph_1}2$ let $A_\eta = \{a_{\xi \restriction \alpha} \mid \alpha < \aleph_1\}$, hence if $\eta \neq \zeta$, then $|A_\eta \cap A_\zeta| \leq \aleph_0$. Suppose by contradiction $\lambda < 2^{\aleph_1}$ and $\{A_i \mid i < \lambda\}$ is a prime family. W.l.o.g. for every $i < \lambda$, A_i contains the set of rational numbers. For every $\eta \in {}^{\aleph_1}2$ let f_η and i_η be such that $f_\eta : A_{i_\eta} \to A_\eta$ is an OP function. Clearly for some $\eta \neq \zeta$, $f_\eta \restriction \mathbb{Q} = f_\zeta \restriction \mathbb{Q}$, and $i_\eta = i_\zeta$. But then f_η differs from f_ζ on at most countably many points. Since $i_\eta = i_\zeta$ and $|A_\eta \cap A_\zeta| \leq \aleph_0$, we reach a contradiction.

(b) Suppose by contradiction that $A \in K$, and it is isomorphic to every element of $P(A)\cap K$. For every $B\in P(A)\cap K$ let f_B be an isomorphism between A and B, we reach a contradiction in a way similar to what was done in (a).

Questions 7.2. (a) Does WCH imply that for every $A\in K$ there is $B\in P(A)\cap K$ such that $A\not\preccurlyeq B$?

(b) Does WCH imply K does not contain a universal element?

Next we want to show that NA is consistent with WCH. This brings up a new application of the club method, which we call the 'nearing forcing'. Recall that A1 is the axiom introduced in Section 6 to replace CH when we want to apply the club method.

Lemma 7.3 (A1). *Let $A, B\in K$. Then there is a c.c.c. forcing set $P=P_{A,B}$ of power $\aleph_1$ such that $\Vdash_P N(A, B)$.*

Proof. Let $M_{A,B}=M$ be a model with universe $\aleph_1$ which encodes $\langle A,<\rangle$ and $\langle B,<\rangle$. Let $N=M^c$ and C be N-thin. We identify $A\cup B$ with $\aleph_1$, hence $A\subseteq\aleph_1$ and $B\subseteq\aleph_1$. Since there are two linear orderings on $\aleph_1$, we denote $a\prec b$ to mean that a is less than b as real numbers, and $a<b$ to mean that a is less than b, as ordinals. Let $\{E_i \mid i<\aleph_1\}$ be an enumeration of the C-slices in an increasing order. Let $P=\{f\in P_{\aleph_0}(A\times B) \mid f$ is an OP function, $\mathrm{Dom}(f)\cup\mathrm{Rng}(f)$ is C-separated, for every $a\in\mathrm{Dom}(f)$ there is $i<\aleph_1$ and $0<N\in\omega$ such that $a\in E_i$ and $f(a)\in E_{i+n}$, and for every distinct $a, b\in\mathrm{Dom}(f)$ if $a<b$ then $f(a)<b\}$. $f\leqslant g$ if $f\subseteq g$. For later use we denote $P=P_{A,B,C}$.

We prove that P is c.c.c. Let $\{f_\alpha \mid \alpha<\aleph_1\}\subseteq P$. W.l.o.g. $f_\alpha=\{\langle a_{\alpha,0}, a_{\alpha,1}\rangle,\ldots,\langle a_{\alpha,2n-2}, a_{\alpha,2n-1}\rangle\}$ where $i<j$ implies $a_{\alpha,i}<a_{\alpha,j}$, there is $m\leqslant n$ such that for every $\alpha<\beta<\aleph_1$, and for every $i<2n$: if $i<2m$, then $a_{\alpha,i}=a_{\beta,i}$, and if $2m\leqslant i$, then $a_{\alpha,i}<a_{\beta,m}$. In addition we can assume that there are pairwise disjoint closed intervals $V_0,\ldots,V_{2n-1}$ such that for every $\alpha<\aleph_1$ and $i<2n$, $a_{\alpha,i}\in V_i$.

We regard each f_α as an element in $(A\times B)^n$. Let F be the topological closure of $\{f_\alpha \mid \alpha<\aleph_1\}$ in $(A\times B)^n$. Let $p\in|N|$ be such that F is definable from p, and every rational interval is definable from p. Let $\gamma_0\in C$ be such that $C\cap[\gamma_0,\aleph_1)\subseteq C_{N,p}$. Let f_α be such that $\gamma_0\leqslant a_{\alpha,2m}$. We denote $a_{\alpha,i}$ by a_i.

We will next duplicate f_α. The new element in the duplication argument is the use of the following fact. If $a_1\neq a_2$, $b_1\neq b_2$ are real numbers, then either $\{\langle a_1, b_1\rangle,\langle a_2, b_2\rangle\}$ or $\{\langle a_1, b_2\rangle,\langle a_2, b_1\rangle\}$ is an OP function. This fact replaces the assumption that X does not contain uncountable 0-colored sets in the proof of SOCA, and the need to preassign colors in the proof of OCA.

We define by a downward induction formulas $\varphi_{2n},\varphi_{2n-1},\ldots,\varphi_m$, and we assume by induction that for every $i: M\vDash\varphi_i[a_0,\ldots,a_{i-1}]$, and that the only parameters of φ_i is p. Let $\varphi_0\equiv\langle x_0,\ldots,x_{2n-1}\rangle\in F$. Suppose φ_{i+1} has been defined.

Since $M \vDash \varphi_{i+1}[a_0, \ldots, a_i]$,

$$M \vDash (\exists x_{i,0}, x_{i,1})\left(x_{i,0} \neq x_{i,1} \wedge \bigwedge_{j=0}^{1} \varphi_{i+1}(a_0, \ldots, a_{i-1}, x_{i,j})\right).$$

Hence there are disjoint rational intervals $U_{i,0}$, $U_{i,1}$ such that

$$M \vDash \bigwedge_{j=0}^{1} (\exists x_{i,j} \in U_{i,j})\ \varphi_{i+1}(a_0, \ldots, a_{i-1}, x_{i,j}).$$

Let

$$\varphi_i \equiv \bigwedge_{j=0}^{1} (\exists x_{i,j} \in U_{i,j})\ \varphi_{i+1}(x_0, \ldots, x_{i-1}, x_{i,j}).$$

Let $m \leqslant i < n$, and let us consider the intervals $U_{2i,0}$, $U_{2i,1}$, $U_{2i+1,0}$, $U_{2i+1,1}$. Let $\varepsilon_i \in \{0, 1\}$ be such that whenever $\langle a_1, b_1\rangle \in U_{2i,0} \times U_{2i+1,\varepsilon_i}$ and $\langle a_2, b_2\rangle \in U_{2i,1} \times U_{2i+1,i-\varepsilon_i}$, then $\{\langle a_1, b_1\rangle, \langle a_2, b_2\rangle\}$ is an OP function. Using $\varphi_{2m}, \ldots, \varphi_{2n}$ we can inductively choose sequences $\langle a^j_{2m}, \ldots, a^j_{2n-1}\rangle$, $j = 0, 1$, such that $\langle a_0, \ldots, a_{2m-1}, a^j_{2m}, \ldots, a^j_{2n-1}\rangle \in F$ and for every $m \leqslant i < n$,

$$a^0_{2i} \in U_{2i,0}, \quad a^0_{2i+1} \in U_{2i+1,\varepsilon_i}, \quad a^1_{2i} \in U_{2i,1} \quad \text{and} \quad a^1_{2i+1} \in U_{2i+1,1-\varepsilon_i}.$$

Since F is the closure of $\{f_\alpha \mid \alpha < \aleph_1\}$, there are β and γ such that for every $m \leqslant i < n$,

$$a_{\beta,2i} \in U_{2i,0}, \quad a_{\beta,2i+1} \in U_{2i+1,\varepsilon_i}, \quad a_{\gamma,2i} \in U_{2i,1} \quad \text{and} \quad a_{\gamma,2i+1} \in U_{2i+1,1-\varepsilon_i}.$$

It follows that $f_\beta \cup f_\gamma \in P$. We have thus proved that P is c.c.c. □

Corollary 7.4. NA + OCA *is consistent.*

Proof. Combine the methods of 3.2 and 7.3. □

Let $\text{cov}(\lambda, \kappa)$ mean that $\kappa < \lambda$, and there is a family $D \subseteq P_{\kappa^+}(\lambda)$ such that $|D| = \lambda$ and for every $c \in P_{\kappa^+}(\lambda)$ there is $d \in D$ such that $c \subseteq d$.

Theorem 7.4. *Let* $V \vDash \text{CH}$, *and let* λ *be a regular cardinal in* V *such that* $\lambda < 2^{\aleph_1} = \mu$ *and* $\text{cov}(\lambda, \aleph_1)$ *holds. Then there is a forcing set P such that* $\Vdash_P \text{NA} + (2^{\aleph_0} = \lambda) + (2^{\aleph_1} = \mu)$.

Remark. Clearly $\text{cov}(\aleph_2, \aleph_1)$ holds, hence one can start with the universe L, and construct V in which $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_3$, and then by 7.3 get a universe satisfying $\text{NA} + (2^{\aleph_0} = \aleph_2) + (2^{\aleph_1} = \aleph_3)$.

Proof. Let V, λ, μ be as in the theorem. Let $P = P_{\text{Cb}}(\lambda)$, $G \subseteq P$ be a generic filter and $W = V[G]$. Using the fact that P is $\aleph_2$-c.c. and does not collapse $\aleph_1$, it is easy to see that $W \vDash \text{cov}(\lambda, \aleph_1)$.

Let $W_\alpha = V[G \cap P_{\text{Cb}}(\alpha)]$, and C_α be the $(\alpha + 1)$st club which is added to V be

$P_{Cb}(\lambda)$. Let $\{\tau_\alpha \mid \alpha<\lambda\}\subset P_{\aleph_2}(\lambda\times\lambda)$ be such that for every $\sigma\in P_{\aleph_2}(\lambda\times\lambda)$, $\{\alpha \mid \sigma\subseteq\tau_\alpha\}$ is unbouned in λ. If R is a forcing set and $R\subseteq\lambda$, then each τ_α can be regarded as an R-name of subset of λ of power $\leqslant\aleph_1$.

We define by induction on $\alpha<\lambda$ a finite support iteration $\langle\{R_\alpha \mid \alpha\leqslant\lambda\}, \{\rho_\alpha \mid \alpha<\lambda\}\rangle$, $\{h_\alpha \mid \alpha<\lambda\}$ and $\{\hat{g}_\alpha \mid \alpha<\lambda\}$ such that: (a) for every α, $|R_\alpha|<\lambda$ and R_α is c.c.c.; (b) h_α is a 1-1 function from R_α into λ, and if $\alpha<\beta$, then $h_\alpha\subseteq h_\beta$; we denote by Q_α the forcing set which is the image of R_α under h_α; (c) for some $\gamma<\lambda$, $\hat{g}_\alpha\in W_\gamma$ and $\Vdash_{Q_\alpha}$ "$\hat{g}_\alpha$ is a 1-1 function from a subset of λ onto $\mathbb{R}$, and if $\alpha<\beta$, then $\hat{g}_\alpha\subseteq\hat{g}_\beta$".

If h is an isomorphism between forcing sets S_1 and S_2, and τ is an S_1-name, let $h(\tau)$ denote the image of τ under h, hence $h(\tau)$ is an S_2-name.

Let R_Q be a trivial forcing set. For a limit ordinal δ let $R_\delta=\bigcup_{\alpha<\delta}R_\alpha$ and $h_\delta=\bigcup_{\alpha<\delta}h_\alpha$. There is some $\gamma<\lambda$ such that $Q_\delta\in W_\gamma$, hence if 5.3, if H is a Q-generic filter, then $\mathbb{R}^{W[H]}=\mathbb{R}^{W_\gamma[H]}$. It thus follows that there is $\hat{g}_\delta\in W_\gamma$ such that $\Vdash_Q$ "$\hat{g}_\delta$ is a 1-1 function from a subset of λ onto R, and for every $\alpha<\delta$, $\hat{g}_\alpha\subseteq\hat{g}$".

Suppose R_α, h_α, $\hat{g}_\alpha$ have been defined, and we define p_α. Let $\hat{g}_\alpha\in W_{\gamma_0}$. Since $|Q_\alpha|, |\tau_\alpha|<\lambda$ and $P_{Cb}(\lambda)$ is $\aleph_2$-c.c., there is $\gamma_1<\lambda$ such that $Q_\alpha, \tau_\alpha\in W_{\gamma_1}$. Let $\gamma\overset{\text{def}}{=}\gamma_\alpha=\gamma_0\cup\gamma_1\cup\alpha$.

In W_γ, $\Vdash_{Q_\alpha}$ "$\hat{g}_\alpha(\tau_\alpha)\in P_{\aleph_2}(\mathbb{R})$", and hence there is a Q_α-name $\tau'_\alpha\in W_\gamma$ such that $\tau_\alpha\subseteq\tau'_\alpha$ and in W_γ, $\Vdash_{Q_\alpha} g_\alpha(\tau'_\alpha)\in K$. By the same argument as in 5.4, if H is Q_α-generic over W and $A=v_H(\hat{g}_\alpha(\tau'_\alpha))$, then C_γ is $M^c_{A,A}$-thin. (Recall that $M_{A,A}$ is a model with universe $\aleph_1$ which encodes $\langle A,<\rangle$.) Hence there is a Q_α-name ρ'_α such that $|\rho'_\alpha|=\aleph_1$ and $\Vdash_{Q_\alpha}\rho'_\alpha=P_{\tau'_\alpha,\tau'_\alpha,C_\gamma}$. (The notation $P_{A,B,C}$ was introduced in the beginning of the proof of 7.3). Let $\rho_\alpha=h_\alpha^{-1}(\rho'_\alpha)$, hence ρ_α is an R_α-name of a forcing set. Let $R_{\alpha+1}=R_\alpha*\rho_\alpha$ and $h_{\alpha+1}$ be a 1-1 function from a subset of λ onto $R_{\alpha+1}$, and $h_{\alpha+1}\supseteq h_\alpha$. $\hat{g}_{\alpha+1}$ is defined as in limit case.

Let H be an R_λ-generic filter, and $H'=h_\lambda(H)$. Let $U=W[H]$ and for every $\alpha\leqslant\lambda$, $g_\alpha=v_{H'}(\hat{g}_\alpha)$. Clearly g_λ is a 1-1 function from a subset of λ onto $\mathbb{R}^U$ and $g_\lambda=\bigcup_{\alpha<\lambda}g_\alpha$. Let $A_1, A_2\in K^U$, $\sigma_i=g_\lambda^{-1}(A_i)$, $i=1,2$, and τ^i be Q_λ-names such that $v_{h(H)}(\tau^i)=\sigma_i$. τ^i can be regarded as an element of $P_{\aleph_2}(\lambda\times\lambda)$ by identifying τ^i with the set $\{\langle\alpha,\beta\rangle \mid \alpha\Vdash_{Q_\lambda}\hat{\beta}\in\tau^i\}$ where $\hat{\beta}$ is the canonical name for β. Let $\alpha<\lambda$ be such that: $\tau^1\cup\tau^2\subseteq\tau_\alpha$, $\tau^1, \tau^2\in W_\alpha$ and $\sigma_1\cup\sigma_2\subseteq\mathrm{Dom}(g_\alpha)$. Let $A=v_{H'}(\hat{g}_\alpha(\tau'_\alpha))$. Hence

$$A\supseteq v_{H'}(\hat{g}_\alpha(\tau_\alpha))=g_\alpha(v_{H'}(\tau_\alpha))\supseteq g_\alpha(\sigma_1\cup\sigma_2)=g_\lambda(\sigma_1\cup\sigma_2)=A_1\cup A_2.$$

$$v_{H\cap R_\alpha}(\rho_\alpha)=v_{H'\cap Q_\alpha}(\rho'_\alpha)=P_{A,A,C_{\gamma_\alpha}}.$$

Q_α, $\hat{g}_\alpha$, $\tau_1, \tau_2\in W_{\gamma_\alpha}$, hence $A_i\in W_{\gamma_\alpha}[H'\cap Q]$. Let M_1 be the expansion of $M_{A,A}$ in which A_1 and A_2 are represented by some unary predicates. Clearly $M_1\in W_{\gamma_\alpha}[H'\cap Q]$, hence C_{γ_α} is M_1-thin. It thus follows that for every C_{γ_α}-slice E, $E\cap A_i$ is dense in A_i, $i=1,2$; so for every ${}^opkv_1$, $D_\beta\overset{\text{def}}{=}\{f\in v_H(\rho_\alpha) \mid$ there is $a>\beta$ such that $a\in\mathrm{Dom}(f)\cap A_1$ and $f(a)\in A_2\}$ is dense in $v_H(\rho_\alpha)$. Let $f_\alpha=\bigcup\{v_H(\tau) \mid \tau\in\rho_\alpha$ and for some $p\in R_\alpha$, $\langle p,\tau\rangle\in H\}$, hence f_α is an OP function from A to A, and by the denseness of the D_β's, $f_\alpha\cap A_1\times A_2$ is uncountable.

We have seen that $U \vDash (\forall A_1, A_2 \in K)\, N(A_1, A_2)$. It remains to show $U \vDash 2^{\aleph_0} = \lambda \wedge 2^{\aleph_1} = \mu$. Since $V \vDash 2^{\aleph_1} = \mu$, $|P_{Cb}(\lambda) * \hat{R}_\lambda| = \mu$, it does not collapse $\aleph_1$ and is $\aleph_2$-c.c.c. $2^{\aleph_1} = \mu$ holds in U too. Since $\mathrm{cov}(\lambda, \aleph_1)$ holds in W, then $W \vDash \lambda^{\aleph_0} = \lambda \cdot \aleph_1^{\aleph_0} = \lambda$; $|R_\lambda| = \lambda$, hence $U \vDash 2^{\aleph_0} \leqslant \lambda$. It is easy to see that for every $\alpha < \beta$, $W[H \cap R_\beta] - W[H \cap R_\alpha]$ contains a real, hence $U \vDash 2^{\aleph_0} = \lambda$. □

Let $A, B \in K$. We say that A and B are *densely near to one another* ($\mathrm{DN}(A, B)$), if there is an OP function $f \subseteq A \times B$ such that $\mathrm{Dom}(f)$, $\mathrm{Rng}(f) \in K$, and $\mathrm{Dom}(f)$, $\mathrm{Rng}(f)$ are dense in A and B respectively. Let $\mathrm{DNA} = (\forall A, B \in K)\, \mathrm{DN}(A, B)$.

Theorem 7.5. NA$\Rightarrow$DNA.

We need some lemmas and terminology. Let $a, b \in {}^{\omega}2$, $a \lessdot b$ if there is $n_0 \in \omega$ such that for every $n \geqslant n_0$, $a(n) \leqslant b(n)$, and $\{n \mid a(n) < b(n)\}$ is infinite. An element of ${}^{\omega}2$ which is not eventually zero is indentified with the real number in the interval $(0, 1]$ which it represents. $\aleph_1 + \aleph_1^*$ denotes the linear ordering which is the sum of $\langle \aleph_1, < \rangle$ and $\langle \aleph_1, > \rangle$.

Hausdorff using ZFC only constructed a sequence $\{a_i \mid i \in \aleph_1 + \aleph_1^*\} \subseteq {}^{\omega}2$ such that: (a) if $i < j \in \aleph_1 + \aleph_1^*$, then $a_i \lessdot a_j$; and (b) there is no $a \in {}^{\omega}2$ such that for every $i \in \aleph_1$ and $j \in \aleph_1^*$, $a_i \lessdot b \lessdot a_j$. We call such a set a Hausdorff set.

Proposition 7.6. (a) *Let $A \subseteq \mathbb{R}$ be a Hausdorff set, and let $B \subseteq \mathbb{R}$ be countable. Then there is a G_δ-set G such that $G \supseteq B$ and $G \cap (A - B) = \emptyset$.*

Proof. Easy and well known. □

Proposition 7.7 (NA). *Let $A \subseteq \mathbb{R}$ be uncountable. Then for every countable $B \subseteq \mathbb{R}$ there is an open set U such that $U \supseteq B$ and $|A - U| \geqslant \aleph_1$.*

Proof. Let A, B be as above, and let C be a Hausdorff set. Let $f \subseteq C \times A$ be an uncountable OP function, let F be the closure of f in $\mathbb{R} \times \mathbb{R}$, and let $D = \bigcup \{F^{-1}(b) \mid b \in B\}$. Clearly for every $b \in R$, $|F^{-1}(b)|, |F(b)| \leqslant 2$, and hence D is countable. Let V be an open set in $\mathbb{R}$ such that $V \supseteq D$ and $|\mathrm{Dom}(f) - V| \geqslant \aleph_1$. W.l.o.g. if $b \in B$ and $F^{-1}(b) = \{d_1, d_2\}$, then V contains the closed interval determined by d_1 and d_2. (This can be assumed since the above interval does not intersect $\mathrm{Dom}(f) - D$. Let $\{V_i \mid i \in \omega\}$ be the partition of V into pairwise disjoint open intervals. We can further assume that the endpoints of each V_i belong to $\mathrm{cl}(\mathrm{Dom}(f))$. (This is so since every open interval I is contained in an open interval J with endpoints in $\mathrm{cl}(\mathrm{Dom}(f))$ such that $I \cap \mathrm{cl}(\mathrm{Dom}(f)) = J \cap \mathrm{cl}(\mathrm{Dom}(f))$.) Let $V_i = (c_i, d_i)$, let $a_i = \min(F(c_i))$, $b_i = \max(F(d_i))$ and $U_i = (a_i, b_i)$. It is easy to check that $U \overset{\text{def}}{=} \bigcup_{i \in \omega} U_i \supseteq B$ and $|\mathrm{Rng}(f) - U| \geqslant \aleph_1$. □

Proof of Theorem 7.5. Assume NA, and let $A, B \in K$. W.l.o.g. A, B are dense in $\mathbb{R}$. Hence we have to construct an OP function $f \subseteq A \times B$ such that $\mathrm{Dom}(f)$, $\mathrm{Rng}(f)$ are dense in $\mathbb{R}$ and belong to K.

We call an OP function f extendible, if $\mathrm{Dom}(f)$ belongs to K and, the closure of f in $\mathbb{R}\times\mathbb{R}$ is an OP function.

Let us first see that if $A, B\in K$, then there is an extendible function $f\subseteq A\times B$. Let $g\subseteq A\times B$ be an uncountable OP function. Let G be the closure of g in $\mathbb{R}\times\mathbb{R}$. Let $C=\{a\in\mathbb{R}\mid \text{there is } a'\in\mathbb{R} \text{ such that } a'\neq a \text{ and } G(a)\cap G(a')\neq\emptyset\}$, and $D=\{a\in\mathbb{R}\mid \text{there is } a'\in\mathbb{R} \text{ such that } a'\neq a \text{ and } G^{-1}(a)\cap G^{-1}(a')\neq\emptyset\}$. Obviously C and D are countable. Let V be an open set containing C such that $\mathrm{Dom}(g)-V$ is uncountable, and let $g_1=g\restriction(\mathrm{Dom}(g)-V)$. Let U be an open set containing D such that $\mathrm{Rng}(g_1)-U$ is uncountable, and let $f_1=g_1\restriction(\mathrm{Dom}(g_1)-g_1^{-1}(U))$. It is easy to check that f_1 is uncountable and its closure is an OP function. Let f be a restriction of f_1 to an element of K, then f is as desired. This concludes the proof of the above claim.

Let $\{I_i\mid i\in\omega\}$ be a list of all rational intervals. We define by induction a sequence of extendible functions $\{f_i\mid i\in\omega\}$ where $f_i\subseteq A\times B$. Let $f_0\subseteq A\times B$ be any extendible function. Suppose f_i has been defined. If $|\mathrm{Dom}(f_i)\cap I_i|\geq 2$, let $f_i'=f_i$. Suppose otherwise, then using the fact that f_i is extendible it is easy to see that there are non-empty open intervals J_1, J_2 such that: $J_1\subseteq I_i$, $J_1\cap\mathrm{Dom}(f_i)=\emptyset$, and for every $a\in\mathrm{Dom}(f_i)$: if $a<J_1$, then $f_i(a)<J_2$, and if $J_1<a$, then $J_2<f_i(a)$. Let $g\subseteq(A\cap J_1)\times(B\cap J_2)$ be an extendible function such that the endpoints of J_1 and J_2 do not belong to $\mathrm{Dom}(g)$ and $\mathrm{Rng}(g)$ respectively. Let $f_i'=f_i\cup g$. It is easy to see that f_i' is extendible.

We define f_{i+1} from f_i' analogously in order to assure that $|\mathrm{Rng}(f_{i+1})\cap I_i|\geq\aleph_1$.

It is easy to see that $f\stackrel{\mathrm{def}}{=}\bigcup_{i\in\omega}f_i$ is an OP function such that $f\subseteq A\times B$ and $\mathrm{Dom}(f)$, $\mathrm{Rng}(f)$ belong to K and are dense in $\mathbb{R}$. □

Question 7.8. Let $\mathrm{NA}^-\equiv(\forall A,B\in K)\ (N(A,B)\cup N(A,B^*))$. Does NA^- imply that $(\forall A,B\in K)\ (\exists I,J)\ (I \text{ and } J \text{ are intervals } \wedge(\mathrm{DN}(A\cap I,B\cap J)\vee \mathrm{DN}(A\cap I,(B\cap J)^*)))$?

8. A weak form of Martin's axiom, the consistency of the incompactness of the Magidor-Malitz quantifiers

In this section we deal with two separate problems. The first is to construct a model of set theory in which the Magidor–Malitz quantifier is countably incompact. This question was raised by Malitz. It was first solved by Shelah (unpublished) using methods of Avraham. The first solution involved properties of Suslin trees which are expressible by sentences in the Magidor–Malitz language (MML). So it was possible to show that CH did not imply the countable compactness of MML. On the other hand, the solution that we present here shows that MA+ $(\aleph_1+2^{\aleph_0})$ does not imply the countable compactness of MML.

The second question we are concerned with is whether the axioms like BA, NA, OCA etc. imply $\mathrm{MA}_{\aleph_1}$. The answer to this question is negative; in fact, forcing sets constructed with the aid of the club method do not destroy Suslin trees. We will formulate some strong form of the chain condition which preserves Suslin trees but can still change the satisfaction of sentences in MML.

We start with the incompactness of MML.

In [1] the notion of a k-entangled set of real numbers was introduced, and it was proved (Theorem 6) that for every $k>0$ $\mathrm{MA}_{\aleph_1}$ + "There exists a k-entangled set" is consistent.

We will use a similar construction in order to get a model of ZFC in which MML is not countably compact. (Using c.c.c.-indestructible S-spaces, K. Kunen showed that the incompactness of MML holds in every model of MA obtained from a ground model satisfying CH by a c.c.c. forcing.)

Let us recall some definitions. A k-place configuration is a sequence $\boldsymbol{\varepsilon} = \langle \varepsilon_0, \ldots, \varepsilon_{k-1} \rangle$ of zeroes and ones. Let $\boldsymbol{a}$, $\boldsymbol{b}$ be sequences of real numbers of length k. $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ has configuration $\boldsymbol{\varepsilon}$ ($\vDash \boldsymbol{\varepsilon}[\boldsymbol{a}, \boldsymbol{b}]$) if for every $i<k$: if $\varepsilon_i = 0$, then $a_i < b_i$, and if $\varepsilon_i = 1$, then $b_i < a_i$.

Definition 8.1 (Shelah [1]). Let $A \subseteq \mathbb{R}$ and $|A| = \aleph_1$. A is *k-entangled*, if for every sequence $\{\boldsymbol{a}_i \mid i<\aleph_1\} \subseteq A^k$ of 1-1 pairwise disjoint sequences, and for every k-place configuration $\boldsymbol{\varepsilon}$ there are $i, j < \aleph_1$ such that $\vDash \boldsymbol{\varepsilon}[\boldsymbol{a}_i, \boldsymbol{a}_j]$.

Note that if A is k-entangled, then it is l-entangled for every $l \leqslant k$.

The following easy claim appears in [1].

Proposition 8.2. $\mathrm{MA}_{\aleph_1}$ *implies that there is no A such that A is k-entangled for every $k>0$.*

The incompactness of MML follows from 8.2 and the following main theorem.

Theorem 8.3. $\mathrm{MA}_{\aleph_1} + (\forall k>0)(\exists A)$ *(A is k-entangled) is consistent.*

Let V be a model of ZFC which satisfies the axiom mentioned in 8.3. We show that MML is countably incompact in V.

Let L be a language containing unary predicates $\{P_i \mid i \in \omega\}$, for every $n>0$ on $(n+1)$-place predicate R_n, a unary predicate Q and a binary predicate $<$. Let T be the following theory in MML.

(1) P_0 is uncountable; $<$ is a linear ordering of P_0.

(2) $Q \subseteq P_0$ and is a countable dense subset relative to $<$.

(3) For $n>0$, R_n is a 1-1 function from P_0^n onto P_n (i.e. P_n represents the set of n-tuples from P_0).

(4) $\langle P_0, < \rangle$ is k-entangled for every $k>0$. (The reader should check that (4) can be expressed by an MML-theory.)

By 8.2, $\mathrm{MA}_{\aleph_1}$ implies that T is not consistent but since for every $k \in \omega$ there is a k-entangled set, every finite subset of T is consistent.

We now proceed to the proof of 8.3.

Definition 8.4. (a) Let $\boldsymbol{k} = \langle k_0, \ldots, k_{n-1} \rangle$ be a sequence of positive natural numbers, $k = \sum_{i<n} k_i$, and $\mathbf{A} = \langle A_0, \ldots, A_{n-1} \rangle$ be a sequence of uncountable subsets of $\mathbb{R}$. We say that $\mathbf{A}$ is $\boldsymbol{k}$-entangled if for every sequence $\{\boldsymbol{a}_i \in i < \aleph_1\} \subseteq A_0^{k_0} \times \cdots \times A_{n-1}^{k_{n-1}}$ of pairwise disjoint 1-1 sequences, and every k-place configuration $\boldsymbol{\varepsilon}$, there are $i, j < \aleph_1$ such that $\models \boldsymbol{\varepsilon}[\boldsymbol{a}_i, \boldsymbol{a}_j]$.

(b) Let $\boldsymbol{k} = \{k_i \mid i \in \omega\}$ be a sequence of positive natural numbers and $\mathbf{A} = \{A_i \mid i \in \omega\}$ be a sequence of uncountable subsets of $\mathbb{R}$. $\mathbf{A}$ is $\boldsymbol{k}$-entangled if for every $n \in \omega$, $\mathbf{A} \restriction n$ is $\boldsymbol{k} \restriction n$-entangled.

The exact form of 8.3 that we prove is the following.

Lemma 8.5. *Suppose $V \models \mathrm{CH} + (2^{\aleph_1} = \aleph_2)$, and suppose $\mathbf{A}$ is an ω-sequence such that $\mathbf{A}$ is $\boldsymbol{k}$-entangled. Then there is a c.c.c. forcing set P of power $\aleph_2$ such that $\Vdash_P$ "$\mathbf{A}$ is $\boldsymbol{k}$-entangled, $2^{\aleph_0} = \aleph_2$ and MA holds".*

Proof. The general framework is the usual one, and we will not repeat it. In each atomic step of the iteration we will apply the explcit contradiction method introduced in Section 2, that is, we are given a c.c.c. forcing set Q of power $\aleph_1$, if $\Vdash_Q$ "$\mathbf{A}$ is $\boldsymbol{k}$-entangled", then Q is the next stage in the iteration. If however there is some $q \in Q$ which forces that $\mathbf{A}$ is not $\boldsymbol{k}$-entangled, then we devise a c.c.c. forcing set R of power $\aleph_1$ such that $\Vdash_R$ "Q is not c.c.c. and $\mathbf{A}$ is $\boldsymbol{k}$-entangled", and add R as the next stage in the iteration. Hence the central claim in the proof is the following.

Claim 1 (CH). *Let $\mathbf{A}$ be $\boldsymbol{k}$-entangled, let Q be a c.c.c. forcing set, $q \in Q$ and $q \Vdash_Q$ "$\mathbf{A}$ is not $\boldsymbol{k}$-entangled". Then there is a c.c.c. forcing set R of power $\aleph_1$ such that $\models_Q$ "Q is not c.c.c. and $\mathbf{A}$ is $\boldsymbol{k}$-entangled".*

Proof. The proof is very similar to the proof of Theorem 6 in [1]. However technically the present proof is somewhat more complicated.

Let $\mathbf{A}$, $\boldsymbol{k}$, Q and q be as in the lemma. Let M be a model with universe $\aleph_1$ which encodes $\mathbf{A}$ and enough set theory. Let $<$ denote the linear ordering of ordinals in M, and $\prec$ denote the linear ordering of real numbers in M. By replacing q by a more informative condition we can w.l.o.g. assume that there is an $n \in \omega$, and n-place configuration $\boldsymbol{\rho} = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ and a Q-name τ such that $q \Vdash_Q$ "τ is an uncountable set of pairwise disjoint 1-1 sequences from $\prod_{i<n} A_i^{k_i}$, and for every $\boldsymbol{a}, \boldsymbol{b} \in \tau$, $\not\models \rho[\boldsymbol{a}, \boldsymbol{b}]$".

Let $k = \sum_{i<n} k_i$. It is easy to construct a sequence $\{\langle q_\alpha, \boldsymbol{a}(\alpha, 0), \ldots, \boldsymbol{a}(\alpha, k) \rangle \mid \alpha < \aleph_1\}$ such that: (1) for every α, $q_\alpha \geq q$; (2) for every α and j, $q_\alpha \Vdash_Q \boldsymbol{a}(\alpha, j) \in \tau$; (3) $\{\boldsymbol{a}(\alpha, j) \mid \alpha < \aleph_1, j \leq k\}$ is a family of pairwise disjoint 1-1 sequences; and (4) for every $\alpha < \aleph_1$, there are ordinals $\beta(\alpha, 0) < \cdots < \beta(\alpha, k)$ in C_M such that: for every $j \leq k$, $\beta(\alpha, j) \leq \boldsymbol{a}(\alpha, j)$, for every $j < k$, $\boldsymbol{a}(\alpha, j) < \beta(\alpha, j+1)$, and for every $\alpha' < \alpha$, $\boldsymbol{a}(\alpha', k) < \beta(\alpha, 0)$. ($\boldsymbol{a} < \beta$ means that all the elements of $\boldsymbol{\alpha}$ are less than β etc.)

Let $\boldsymbol{a}(\alpha)=\boldsymbol{a}(\alpha,0)^\frown\cdots{}^\frown a(\alpha,k)$, and let $\boldsymbol{a}(\alpha,j)=\langle a(\alpha,j,0),\ldots,a(\alpha,j,k-1)\rangle$. By further uniformization we can assume that there is a family of pairwise disjoint open rational intervals $\{U(j,i)\mid j\leqslant k, i<k\}$ such that for every α, j, i, $a(\alpha,j,i)\in U(j,i)$.

We say that q_α and q_β are explicitly contradictory if for some $j\leqslant k$, $\vDash\boldsymbol{\rho}[\boldsymbol{a}(\alpha,j),\boldsymbol{a}(\beta,j)]$ or $\vDash[\boldsymbol{a}(\beta,j),\boldsymbol{a}(\alpha,j)]$. Clearly if q_α and q_β are explicitly contradictory, then they are incompatible in Q. Let $R=\{\sigma\in P_{\aleph_0}(\aleph_1)\mid$ for every $\alpha\neq\beta\in\sigma$, q_α and q_β are explicitly contradictory$\}$. $\sigma\leqslant\eta$ if $\sigma\subseteq\eta$.

Clearly if G is R-generic and $A=\{q_\alpha\mid\{\alpha\}\in G\}$, then A is an antichain in Q. Once we show that R is c.c.c., there is a standard argument which shows that there is $r_0\in R$ such that $r_0\Vdash_R$ "$\hat{A}$ is uncountable" where $\hat{A}$ is the standard name for $\{q_\alpha\mid\{\alpha\}\in G\}$. Hence by replacing R by $R'\overset{\text{def}}{=}\{r\in R\mid r\geqslant r_0\}$ we can conclude that $\Vdash_{R'}$ "Q is not c.c.c.".

The proof that $\Vdash_R$ "$\boldsymbol{A}$ is $\boldsymbol{k}$-entangled" already includes the arguments appearing in the proof that R is c.c.c., so we omit the latter.

Suppose by contradiction there is $r\in R$ such that $r\Vdash_R$ "$\boldsymbol{A}$ is not $\boldsymbol{k}$-entangled". W.l.o.g. $r=0$, and $m\geqslant n$ is such that $\Vdash_R$ "$\boldsymbol{A}\restriction m$ is not $(\boldsymbol{k}\restriction m)$-entangled". Let $\boldsymbol{\varepsilon}$ be a configuration and η be an R-name such that $\Vdash_R$ "η is an uncountable set of 1-1 pairwise disjoint sequences from $\prod_{i<m}A_i^{k_i}$, and there are no $\boldsymbol{a},\boldsymbol{b}\in\eta$ such that $\vDash\boldsymbol{\varepsilon}[\boldsymbol{a},\boldsymbol{b}]$".

Let $\{\langle r_\alpha,\boldsymbol{b}_\alpha\rangle\mid\alpha<\aleph_1\}$ be such that $r_\alpha\Vdash_R$ "$\boldsymbol{b}_\alpha\in\eta$, and $\{\boldsymbol{b}_\alpha\mid\alpha<\aleph_1\}$ is a family of pairwise disjoint sequences".

We now have to uniformize the sequence $\{\langle r_\alpha,\boldsymbol{b}_\alpha\rangle\mid\alpha<\aleph_1\}$ as much as possible. We do not repeat the details of this process which have already appeared so many times.

Suppose $r_\alpha=\{\gamma(\alpha,0),\ldots,\gamma(\alpha,l-1)\}$ where the γ's appear in an increasing order. Let us assume for simplicity that the r_α's are pairwise disjoint. For every α we form the following sequence $\boldsymbol{c}_\alpha$ which belongs to a finite product of A_i's. Let $\boldsymbol{a}_\alpha=\boldsymbol{a}(\gamma(\alpha,0))^\frown\cdots{}^\frown\boldsymbol{a}(\gamma(\alpha,l-1))$ and $\boldsymbol{c}_\alpha=\boldsymbol{a}_\alpha{}^\frown b_\alpha$. Let $F=\mathrm{cl}(\{\boldsymbol{c}_\alpha\mid\alpha<\aleph_1\})$, and $\delta_0\in C_M$ be such that F is definable in M by some parameter $<\delta_0$. Let α be such that $\delta_0\leqslant\boldsymbol{c}_\alpha$.

We want to duplicate $\langle r_\alpha,\boldsymbol{b}_\alpha\rangle$. Before stating the exact claim we need additional notations. Let $\boldsymbol{d}(i,j)=\boldsymbol{a}(\gamma(\alpha,i),j)$, $\boldsymbol{d}(i)=\boldsymbol{d}(i,0)^\frown\cdots{}^\frown\boldsymbol{d}(i,k)$, and $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ denote respectively $\boldsymbol{a}_\alpha$, $\boldsymbol{b}_\alpha$ and $\boldsymbol{c}_\alpha$.

Duplication claim. *There are $\boldsymbol{c}^0,\boldsymbol{c}^1\in F$ such that*

$$\boldsymbol{c}^i=\boldsymbol{d}^i(0,0)^\frown\cdots{}^\frown\boldsymbol{d}^i(0,k)^\frown\cdots{}^\frown\boldsymbol{d}^i(l-1,0)^\frown\cdots{}^\frown d^i(l-1,k)^\frown\boldsymbol{b}^i,\qquad i=1,2$$

and (1) $\vDash\boldsymbol{\varepsilon}[\boldsymbol{b}^0,\boldsymbol{b}^1]$, (2) *for every $i<l$, there is $j\leqslant k$ such that* $\vDash\boldsymbol{\rho}[\boldsymbol{d}^0(i,j),\boldsymbol{d}^1(i,j)]$.

The contradiction follows easily from the duplication claim by choosing a $\boldsymbol{c}_\beta$

near enough to c^0 and a c_γ near enough to c^1, because for such β and $\gamma: r_\beta \cup r_\gamma \in R$ but $\vDash \varepsilon[b_\beta, b_\gamma]$.

Proof of the duplication claim. Let $<_1$ denote the lexicographic order on $l \times (k+1)$ let $t = l(k+1)$ and let $s_0 <_1 s_1 <_1 \cdots <_1 s_{t-1}$ be an enumeration of $l \times (k+1)$. By the construction, there are $\beta(s_j) = \beta_j$ in C_M such that $\beta_0 \leq d(s_0) < \beta_1 \leq \cdots < \beta_{t-1} \leq d(s_{t-1})$. Let $E(s_j) = E_j = [\beta_j, \beta_{j+1})$. We next divide b and ε into parts in the following way. $b = \langle b_0, \ldots, b_{k-1}, b_k, \ldots, b_\upsilon \rangle$ where $b_0, \ldots, b_{k-1}$ belong to some A_i where $i < n$ and $b_k, \ldots, b_\upsilon$ belong to some A_i where $n \leq i < m$. Let $e(s_j) = e_j$ be the restriction of b to those coordinates w for which $b_w \in E_j \cap \bigcup_{i<n} A_i$, and let f_j be the restriction of b to those coordinates w such that $b_w \in E_j \cap \bigcup_{n \leq i < m} A_i$. Let ε_j be the restriction of ε such that $\mathrm{Dom}(e_j) = \mathrm{Dom}(\varepsilon_j)$, and similarly ρ_j is a restriction of ε such that $\mathrm{Dom}(\rho_j) = \mathrm{Dom}(f_j)$. Let $e = \bigcup_{j<t} e_j$.

By the entangledness of $\mathbf{A}$ we can construct $c^0, c^1 \in F$ having the form

$$c^i = d^i(s_0)^\frown \cdots {}^\frown d^i(s_{t-1})^\frown \Big(\bigcup_{j<t} e_j^i \cup \bigcup_{j<t} f_j^i \Big), \quad i = 1, 2,$$

such that for every $j < t$: if $e_j = \Lambda$, then $\vDash \rho[d^0(s_j), d^1(s_j)]$ and $\vDash \rho_j[f_j^0, f_j^1]$; and if $e_j \neq \Lambda$, then $\vDash \varepsilon_j[e_j^0, e_j^1]$ and $\vDash \rho_j[f_j^0, f_j^1]$.

This is proved by the usual duplication argument, that is, we first define by a downward induction some formulas $\psi_{t-1}, \ldots, \psi_0$, and then starting with ψ_0 we construct by induction on j, $d^i(s_j)$, e_j^i and f_j^i, $i = 1, 2$.

Since e contains only k elements whereas for every i we have $k+1$ $d(i, j)$'s it follows that for every $i < l$, there is j such that $e(i, j) = \Lambda$. Hence for every i, there is j such that $\vDash \rho[d^0(i, j), d^1(i, j)]$. Hence the duplication claim is proved. This concludes the proof of claim 1, and hence the proof of 8.5. □

There is still a gap between 8.3 and 8.5. In order that 8.3 will follow we still need the following easy claim.

Claim. *There is a universe V satisfying* CH *and* $2^{\aleph_1} = \aleph_2$ *such that there is a sequence* $\mathbf{A}$ *in V which is* $\mathbf{k}$*-entangled, where* $\mathbf{k} = \langle 1, 2, 3, \ldots \rangle$.

In fact (CH) implies that there is $A \in K$ such that A is k-entangled for every k, and hence if $\{A_i \mid i \in \omega\}$ is a partition of A into uncountable sets, then $\{A_i \mid i \in \omega\}$ is $\mathbf{k}$-entangled for every $\mathbf{k}$.

However, instead, we can start with a universe V satisfying CH and $2^{\aleph_1} = \aleph_2$, and then add to V a set of $\aleph_1$ Cohen reals. It is shown in [1] that such an A is k-entangled for every k.

We turn now to the proof that Suslin trees are preserved under forcing sets constructed by the club method.

Definition 8.6. Let P be a forcing set. P has the strong countable chain condition (P is s.c.c.) if for every uncountable $A \subseteq P$, there is $B \subseteq A$ and $\{B_{i,j} \mid i \in \omega, j = 0, 1\}$

such that B is uncountable, for every i and j, $B_{i,j} \subseteq B$; and (1) for every $i \in \omega$, $b_0 \in B_{i,0}$ and $b_1 \in B_{i,1}$, b_0 and b_1 are compatible, (2) for every uncountable $C \subseteq B$, there is i such that $B_{i,0} \cap C = \emptyset$ and $B_{i,1} \cap C \neq \emptyset$.

Remarks. Clearly every s.c.c. forcing set is c.c.c. Kunen and Tall [10] defined the following property of forcing sets. P has property S if every uncountable subset of P contains an uncountable subset of pairwise compatible conditions. Clearly $S \Rightarrow$ s.c.c. Let MML_2 be the portion of MML in which only the quantifier Q^2 which bounds two variables is used. If $\varphi \in \mathrm{MML}_2$, $M \models \varphi$ and P has property S, then $\Vdash_P (M \models \varphi)$. If P is s.c.c. this is not longer true, since e.g. $N(A, B)$ can be expressed by an MML_2 sentence about the model $M = \langle A \cup B; <, A, B \rangle$. Now suppse $\neg N(A, B)$ holds in V, and let P the nearing forcing of A and B. Then P is s.c.c., and $\Vdash_P N(A, B)$.

It seems that "P is s.c.c. and $\Vdash_P \pi$ is s.c.c." does not imply that $P * \pi$ is s.c.c. However, since we are dealing with the preservation of MML sentences we can avoid this problem by using the following lemma.

Lemma 8.7. *Let $M \in V$ be a model and $\varphi \in \mathrm{MML}$ be a sentence of the form $\neg Q^n x_1 \cdots x_n R(x_1, \ldots, x_n)$ where R is a relation symbol. Let $\langle \{P_i \mid i \leqslant \alpha\}, \{\pi_i \mid i < \alpha\} \rangle$ be a finite support iteration of c.c.c. forcing sets such that α is a limit ordinal, and for every $\beta < \alpha$, $\Vdash_{P_\beta} (M \models \varphi)$. Then $\Vdash_{P_\alpha} (M \models \varphi)$.*

Proof. Suppose by contradiction that $p \Vdash_{P_\alpha} (M \models \neg\varphi)$. Then there is a P_α-name τ such that $p \Vdash_{P_\alpha}$ "τ is uncountable and every n-tuple from τ satisfies R". Let $\{\langle p_i, a_i \rangle \mid i < \aleph_1\} \subseteq P_\alpha \times |M|$ be such that for every i, $p_i \geqslant p$, if $i \neq j$ then $a_i \neq a_j$, and $p_i \Vdash \hat{a}_i \in \tau$, where $\hat{a}_i$ is the canonical name of a_i. For $q \in P_\alpha$ let sup(q) be the support of q, hence $\sup(q) \in P_{\aleph_0}(\alpha)$. If there is β such that $\sup(p) \subseteq \beta$ and for $\aleph_1$ p_i's, $\sup(p_i) \subseteq \beta$, then $\Vdash_{P_\beta} (M \models \neg\varphi)$, which is a contradiction. Hence the cofinality of α must be $\aleph_1$, and we can w.l.o.g. assume that $\sup(p_i)$ constitutes a Δ-system of the following form: $\sup(p_i) = \sigma \cup \sigma_i$ where for every $i \neq j$, $\sigma_i \cap \sigma_j = \emptyset$, and for every $\beta \in \sigma$ and $\gamma \in \sigma_i$, $\beta < \gamma$. Let β be such that for $\sigma \subseteq \beta$, $\sup(p) \subseteq \beta$ and for every i and $\gamma \in \sigma_i$, $\beta \leqslant \gamma$. Let $\tau' = \{\langle p_i \restriction \beta, \hat{a}_i \rangle \mid i < \aleph_1\}$. τ' is a P_β-name. Clearly for every $i_1, \ldots, i_n < \aleph_1$: there is $q \in P_\alpha$ such that $q \geqslant p_{i_1}, \ldots, p_{i_n}$ iff there is $q \in P_\beta$ such that $q \geqslant p_{i_1} \restriction \beta, \ldots, p_{i_n} \restriction \beta$. It thus follows that for some $p' \in P_\beta$, $p' \Vdash_{P_\beta}$ "τ' is uncountable and every n-tuple from τ' satisfies R". Hence $\Vdash_{P_\beta} (M \models \neg\varphi)$, a contradiction. $\square$

Recall that our aim is to show that Suslin trees are preserved under forcing sets constructed by the club method. Let $\langle T, < \rangle$ be a Suslin tree in V. In order that it will remain a Suslin tree after forcing with some forcing set P it has to satisfy in V^P the following sentence

$$\varphi \equiv \neg Q^2 xy\, ((\neg x < y \wedge \neg y < x) \rightarrow x = y).$$

By the last lemma if $P=P_\alpha$ where $\langle\{P_i \mid i<\alpha\}, \{\pi_i \mid i<\alpha\}\rangle$ is a finite support iteration of c.c.c forcing sets, then it suffices to show that for every $i<\alpha$, $\Vdash_{P_i}$ "If $\langle T, <\rangle$ is Suslin tree, then $\Vdash_{\pi_i} \langle T, <\rangle$ is a Suslin tree". So we prove two claims.

Lemma 8.8. *If P is s.c.c., and $T\in V$ is a Suslin tree, then $\Vdash_P$ "T is a Suslin tree".*

Observation 8.9. *If P is constructed by the club method, then it is s.c.c.*

Observation 8.9 is not an exact mathematical statement since we have never defined what it means for a forcing set P to be constructed with the aid of the club method. However, the reader can easily check that in every case in which we applied the club method the resulting forcing set was indeed s.c.c.

Proof of 8.8. Let P and T be as above and suppose by contradiction that there is $p^0\in P$ such that $p^0 \Vdash_P$ "T is not a Suslin tree". Then there is a P-name τ such that $p^0 \Vdash_P$ "τ is an uncountable antichain in T". Let $\{\langle p_i, a_i\rangle \mid i<\aleph_1\}\subseteq P\times T$ be such that for every $i<\aleph_1$, $p_i \geqslant p^0$, $p_i \Vdash_P \hat{a}_i\in\tau$, and if $j\neq i$, then $a_j\neq a_i$. W.l.o.g. there are $\{B_{ij} \mid i\in\omega, j=0,1\}$ such that $B_{i,j}\subseteq\{p_i \mid i<\aleph_1\}$, for every $q_j\in B_{i,j}$, $j=0,1$, q_0 and q_1 are compatible, and for every uncountable $B\subseteq\{p_i \mid i<\aleph_1\}$ there is $i\in\omega$ such that $B\cap B_{i,j}\neq\emptyset$ for $j=0,1$.

Let $A=\{a_i \mid i<\aleph_1\}$ and $A_{ij}=\{a_\alpha \mid p_\alpha\in B_{i,j}\}$. Hence we have the following situation: (1) A is an uncountable subset of T; (2) if $i\in\omega$ $a^0\in A_{i,0}$ and $a^1\in A_{i,1}$, then a^0 and a^1 are incomparable in T; and (3) if A^1 is an uncountable subset of A, then for some $i\in\omega$, $A^1\cap A_{i,0}$, $A^1\cap A_{i,1}\neq\emptyset$.

We will now show that if T is a Suslin tree, then there are no A and $\{A_{ij} \mid i\in\omega=j=0,1\}$ as above.

Let T be a Suslin tree and $h: T\to P(\omega)$ such that if $a<b$, then $h(a)\subseteq h(b)$. We show that there is $a\in T$ such that for every $b>a$, $h(b)=h(a)$. Suppose not. Let $\{\langle a_i, b_i\rangle \mid i<\aleph_1\}\subseteq T\times T$ be such that for every i, $a_i<b_i$ and $h(a_i)\subsetneq h(b_i)$, and if $i\neq j$, then $a_i\neq a_j$, b_j. W.l.o.g. there is $n\in\omega$ such that for every $i<\aleph_1$, $n\in h(b_i)-h(a_i)$, and if $i<j$, then the level of b_i is less than the level of a_j. Let i and j be such that $b_i<b_j$. Hence $b_i<a_j$, however $n\in h(b_i)$ and $n\notin h(A_j)$, contradicting the monotonicity of h.

Returning to our original claim, suppose by contradiction T is a Suslin tree and A, $\{A_{i,j} \mid i\in\omega, j=0,1\}$ satisfy (1), (2) and (3) above. Since A is a Suslin tree, we can assume that $T=A$. Let $h: T\to P(\omega\times\omega)$ be defined as follows: $h(a)=\{\langle i,j\rangle \mid \text{there is } b\leqslant a \text{ such that } b\in A_{i,j}\}$. Hence, if $b\leqslant a$, then $h(b)\subseteq h(a)$. Let a be such that for every $b>a$, $h(b)=h(a)$, and let $A^1=\{b \mid b>a\}$. Since A^1 is uncountable there are $i\in\omega$ and $b_0, b_1\in A^1$ such that $b_0\in A_{i,0}$ and $b_1\in A_{i,1}$. Hence $\langle i,0\rangle\in h(b_0)$ and $\langle i,1\rangle\in h(b_1)$. Since $a<b_0, b_1$, there follows $\langle i,0\rangle, \langle i,1\rangle\in h(a)$, but this means that there are $a_0, a_1, \leqslant a$ such that $a_j\in A_{i,j}$, $j=1,2$. a_0, a_1 are comparable, and this contradicts (2). Hence the lemmma is proved. □

Let MSA_κ be the axiom saying that for every s.c.c forcing set P and every family $\{D_i \mid i<\kappa\}$ of dense subsets of P, there is a filter of P intersecting every D_i.

Corollary 8.10. (a) $\neg$SH + MSA$_{\aleph_\alpha}$ *is consistent.*
(b) $\neg$SH + TCA *is consistent.*

Proof. (a) follows from 8.7 and 8.9, and (b) follows from 8.7, 8.8, 8.9. In fact in (b) one can replace TCA by any consistent conjunction of axioms whose consistency was proved by the club method. Also in (b) one can add $2^{\aleph_0} \geqslant \aleph_\alpha$. $\square$

9. The isomorphizing forcing, and more on the possible structure of K

The new tool to be presented in this section is the isomorphizing forcing. Given $A, B \in K$ we construct a forcing set $P_{A,B}$ which makes A and B isomorphic.

Baumgartner [2] constructed a $P_{A,B}$ as above in order to prove the consistency of BA. However, since our construction is more canonical, it is easier to combine it with the other methods we have presented.

In this section we use $P_{A,B}$ in order to get (seemingly) a strengthening of BA. by combining the new technique with other methods we obtain a variety of consistency results on the structure of K.

Let $\mathscr{E}$ be a partition of $\aleph_1$ and $\sigma \subseteq \aleph_1 \times \aleph_1$. We define the graph $G_\sigma^{\mathscr{E}}$. The set of vertices of the graph $V_\sigma^{\mathscr{E}}$ is $\{E \in \mathscr{E} \mid (\exists a, b)\, (\langle a, b\rangle \in \sigma \text{ and } (a \in E \text{ or } b \in E))\}$. The set of edges is σ, and E_1, E_2 which belongs to $V_\sigma^{\mathscr{E}}$ are connected by $\langle a, b\rangle \in \sigma$, if $a \in E_1$ and $b \in E_2$ or $b \in E_1$ and $a \in E_2$. When $\mathscr{E}$ is fixed and σ varies, we denote $G_\sigma^{\mathscr{E}}$ and $V_\sigma^{\mathscr{E}}$ by G_σ and V_σ respectively.

We say that a graph G is cycle free it it does not contain cycles, i.e. it does not contain a sequence of vertices $a_1, \ldots, a_n$ and a set of distinct edges $e_1, \ldots, e_n$ such that e_i connects a_i and a_{i+1} and e_n connects a_n and a_1. Let $C \subseteq \aleph_1$ be a club, let $\mathscr{E}^C$ denote the set of C-slices and let $\{E_i^C \mid i < \aleph_1\}$ be an enumeration of $\mathscr{E}$ in an increasing order. We regard the set $E = \{\alpha \mid \alpha < \min(C)$ as a C-slice, hence $E = E_0^C$. E_i^C and E_j^C are near, if for some $n \in \omega$, $i + n = j$ or $j + n = i$. Let $a \in \aleph_1$. $E^C(a)$ denotes the member of $\mathscr{E}^C$ to which a belongs. $E^C(a)$ is abbreviated by $E(a)$ when C is fixed.

Let $\prec$ be a linear ordering of a subset of $\aleph_1$, $C \subseteq \aleph_1$ be a club and $A, B \subseteq \aleph_1$. We define $P = P(C, \prec, A, B)$:

$P = \{f \in P_{\aleph_0}(A \times B) \mid f$ is an OP function with respect to $\prec$, G_f^C is cycle free, and if $f(a) = b$ then $E^C(a)$ and $E^C(b)$ are near$\}$.

$f \leqslant g$ if $f \subseteq g$.

Theorem 9.1. *Let $A, B \in K$, and M be a model such that $|M| \supseteq \aleph_1$. There is a linear ordering $\prec$ on $\aleph_1$ definable in M such that $\langle \aleph_1, \prec\rangle$ is embeddable in $\langle \mathbb{R}, <\rangle$, and $\langle A \cup B, <\rangle$ is embeddable in $\langle \aleph_1, \prec\rangle$, w.l.o.g. $A, B \subseteq \aleph_1$, and $\aleph_1$ and the usual linear ordering $<$ of $\aleph_1$ are definable in M. Let C be M^c-thin, and suppose further that for every C-slice, E, $\langle A \cap E, \prec\rangle$ and $\langle B \cap E, \prec\rangle$ are dense in $\langle A, \prec\rangle$ and $\langle B, \prec\rangle$ respectively. Then* (a) $P = P(C, \prec, A, B)$ *is c.c.c.; and* (b) $\Vdash_P A \cong B$.

Remarks. (a) Baumgartner [2] proved a similar theorem but he used (CH).

(b) Note that we did not have to assume that A and B are definable in M.

Proof. (b) Let $f \in P$ and $a \in A - \mathrm{Dom}(f)$. We show that there is $g \geq f$ such that $a \in \mathrm{Dom}(g)$. V_f is finite, hence there is a C-slice E such that $E \notin V_f$, $E \neq E(a)$, and E and $E(a)$ are near. Since B is dense in itself and $B \cap E$ is dense in B, there is $b \in B \cap E$ such that $g \stackrel{\text{def}}{=} f \cup \{\langle a, b\rangle\}$ is OP. By the choice of E, $g \in P$, hence g is as required. Similarly if $b \in B - \mathrm{Rng}(f)$, then there is $g \in P$ such that $b \in \mathrm{Rng}(g)$. This proves (b).

(a) If V and W are sets of pairs of real numbers we say that $\langle V, W\rangle$ is OP if for every $\langle v_0, v_1\rangle \in V$ and $\langle w_0, w_1\rangle \in W$, $\{\langle v_0, v_1\rangle, \langle w_0, w_1\rangle\}$ is OP. Analogously we define the notion $\langle V, W\rangle$ is order reversing (OR). Note that if U_i, $i = 0, \ldots, 3$, are pairwise disjoint intervals, then $\langle U_0 \times U_1, U_2 \times U_3\rangle$ is either OP or OR.

Let $\{f_\alpha \mid \alpha < \aleph_1\} = F_0 \subseteq P$. As in the previous cases we uniformize F_0 as much as possible. We thus assume that F_0 is a Δ-system, and it will suffice to deal with the case when the kernel of F_0 is empty. Hence let us assume that

$$f_\alpha = \{\langle a(\alpha, 0), a(\alpha, 1)\rangle, \ldots, \langle a(\alpha, 2n-2), a(\alpha, 2n-1)\rangle\}$$

where the $a(\alpha, 2i)$'s are distinct, and if $\alpha < \beta$, $c \in \mathrm{Dom}(f_\alpha) \cup \mathrm{Rng}(f_\alpha)$ and $d \in \mathrm{Dom}(f_\beta) \cup \mathrm{Rng}(f_\beta)$, then $E(c) \neq E(d)$. This last condition assures that if $f_\alpha \cup f_\beta$ is OP, then $f_\alpha \cup f_\beta \in P$.

Let $\boldsymbol{a}(\alpha) = \langle a(\alpha, 0), \ldots, a(\alpha, 2n-1)\rangle$, $F_1 = \{\boldsymbol{a}(\alpha) \mid \alpha < \aleph_1\}$ and F be the topological closure of F_1 in $(\langle \aleph_1, <\rangle)^{2n}$. It will be convenient (however not necessary) to assume that all the $a(\alpha, i)$'s are distinct, hence w.l.o.g. we assume that $A \cap B = \emptyset$. Let $D \in |M^c|$ be such that F is definable from D in M^c, and there is some countable open base of $\langle \aleph_1, < \rangle$ consisting of intervals whose elements are definable from D in M^c. Let $\gamma_0 \in C$ be such that $C \cap [\gamma_0, \aleph_1) \subseteq \{\alpha \mid (\exists N \prec M^c)(D \in |N|$ and $|N| \cap \aleph_1 = \alpha)\}$. Let $f_\alpha = f$ be such that for every $i < 2n$, $\gamma_0 \leq a(\alpha, i)$. We denote $a(\alpha, i)$ by $a(i)$, $\boldsymbol{a}(\alpha) = \boldsymbol{a}$ and $W = \mathrm{Dom}(f_\alpha) \cup \mathrm{Rng}(f_\alpha)$. Let $E^0, \ldots, E^{k-1}$ be the set of C-slices which intersect W, arranged in an increasing order. Let $\boldsymbol{x} = \langle x(0), \ldots, x(2n-1)\rangle$ be a sequence of variables. For every $s < k$, let $R_s = \{i \mid a(i) \in E^s\}$, $\boldsymbol{a}_s = \boldsymbol{a} \restriction R_s$ and $\boldsymbol{x}_s = \boldsymbol{x} \restriction R_s$. Hence $\bigcup_{s<k} \boldsymbol{a}_s = \boldsymbol{a}$ and $\bigcup_{s<k} \boldsymbol{x}_s = \boldsymbol{x}$.

We are now ready for the duplication argument. We define by a downward induction on $s = k, \ldots, 0$ formulas $\rho_s(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{s-1})$ such that the only parameter in ρ_s is D, and $M^c \models \rho_s[\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{s-1}]$. Let $\varphi_k \equiv \bigcup_{s<k} \boldsymbol{x}_s \in F$. Suppose φ_{s+1} has been defined and we define φ_s. Clearly by our assumptions

$$M^c \models (\exists \boldsymbol{x}_s^0, \boldsymbol{x}_s^1)\left(\mathrm{Rng}(\boldsymbol{x}_s^0) \cap \mathrm{Rng}(\boldsymbol{x}_s^1) = \emptyset \wedge \bigwedge_{l=0}^{1} \varphi_{s+1}(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{s-1}, \boldsymbol{x}_s^l)\right).$$

For every $i \in R_s$ and $l = 0, 1$, let U_i^l be an interval definable from D such that: (1) the U_i^l's are pairwise disjoint; (2) if $s' > s$, $i' \in R_{s'}$ and $l' \in \{0, 1\}$, then $U_i^l \cap U_{i'}^{l'} = \emptyset$; and

$$(3) \qquad M^c \models (\exists \boldsymbol{x}_s^0, \boldsymbol{x}_s^1)\left(\bigwedge_{l=0}^{1}\left(\boldsymbol{x}_s \in \prod_{i \in R_s} U_i^l\right) \wedge \bigwedge_{l=0}^{1} \varphi_{s+1}(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{s-1}, \boldsymbol{x}_s^l)\right).$$

Let

$$\varphi_s \equiv (\exists \boldsymbol{x}_s^0, \boldsymbol{x}_s^1)\left(\bigwedge_{l=0}^{1}\left(\boldsymbol{x}_s^l \in \prod_{i\in R_s} U_i^l\right) \wedge \bigwedge_{l=0}^{1} \varphi_{s+1}(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{s-1}, \boldsymbol{x}_s)\right).$$

We prove that we can find for every $s<k$, $l(s)\in\{0,1\}$ such that for every $i<n$: if $2i\in R_s$ and $2i+1\in R_t$, then $\langle U_{2i}^{l(s)}\times U^{l(t)}, U_{2i}^{1-l(s)}\times U_{2i+1}^{1-l(t)}\rangle$ is OP. Recall that the graph G_f has vertices $E^0,\ldots,E^{k-1}$; the edges of G_f are $\langle a(0), a(1)\rangle, \ldots, \langle a(2n-2), a(2n-1)\rangle$; E^s is connected to E^t by $\langle a(2i), a(2i+1)\rangle$ if $a(2i)\in E^s$ and $a(2i+1)\in E^t$ or $a(2i)\in E^t$ and $a(2i+1)\in E^s$; and G_f is cycle free. Let $S\subseteq k$ be such that for every component T of G_f there is a unique $s\in S$ such that $E^s\in T$. Let $S_j=\{s\in k \mid \text{there is } t\in S \text{ and a 1-1 path in } G_f \text{ of length } j \text{ connecting } E^s \text{ with } E^t\}$. Since G_f is cycle free the S_j's are pairwise disjoint and moreover for every $t\in S_{j+1}$ there is a unique edge in G_f which connects E^t with some element of $\{E^s \mid s\in S_j\}$.

For every $s\in S_0$ define $l(s)=0$. Suppose $l(s)$ has been defined for every $s\in\bigcup_{m\leqslant j} S_m$. Let $t\in S_{j+1}$; let $\langle a(2i), a(2i+1)\rangle$ be the unique edge connecting E^t to an element of $\{E^s \mid s\in S_j\}$, and w.l.o.g. suppose that $s\in S_j$, $a(2i)\in E^s$ and $a(2i+1)\in E^t$. Define $l(t)$ in such a way that $\langle U_{2i}^{l(s)}\times U_{2i+1}^{l(t)}, U_{2i}^{1-l(s)}\times U_{2i+1}^{1-l(t)}\rangle$ will be OP. We have defined $l(s)$ for every $s\in k$, and it is easy to check that $\{l(s)\mid s\in k\}$ is as required.

Using the φ_i's we will now construct two members of F. Since $M^c\vDash\varphi_0$, there is $\boldsymbol{b}_0^0$ such that $M^c\vDash b_0^0\in\prod_{i\in R_0} U_i^{l(0)}\wedge\varphi_1(\boldsymbol{b}_0^0)$. Suppose $\boldsymbol{b}_0^0,\ldots,\boldsymbol{b}_{s-1}^0$ have been defined in such a way that $M^c\vDash\varphi_s[\boldsymbol{b}_0^0,\ldots,\boldsymbol{b}_{s-1}^0]$; hence by the definition of φ_s there is $\boldsymbol{b}_s^0$ such that $M^c\vDash\boldsymbol{b}_s^0\in\prod_{i\in R_s} U_i^{l(s)}\wedge\varphi_{s+1}(\boldsymbol{b}_0^0,\ldots,\boldsymbol{b}_s^0)$. According to this definition we obtain $\boldsymbol{b}_0^0,\ldots,\boldsymbol{b}_{k-1}^0$ such that $\bigcup_{s<k} b_s^0\in F$ (this is assured by φ_k) and for every $s<k$, $b_s^0\in\prod_{i\in R_s} U_i^{l(s)}$. Similarly we can define $\boldsymbol{b}_s^1$, $s<k$, such that $\bigcup_{s<k}\boldsymbol{b}_s^1\in F$ and $\boldsymbol{b}_s^1\in\prod_{i\in R_s} U_i^{1-l(s)}$. For $l=0,1$, let $\boldsymbol{b}^l=\bigcup_{s<k}\boldsymbol{b}_s^l$ and $\boldsymbol{b}^l=\langle\boldsymbol{b}^l(0),\ldots,b^l(2n-1)\rangle$. By the construction, for every $i<n$, $\{\langle b^0(2i), b^0(2i+1)\rangle, \langle b^1(2i), b^1(2i+1)\rangle\}$ is OP. Since $F=\mathrm{cl}(\{\boldsymbol{a}(\beta)\mid\beta<\aleph_1\})$, there are β, γ such that for every $s<k$,

$$\boldsymbol{a}(\beta)\restriction R_s\in\prod_{i\in R_s} U_i^{l(s)} \quad\text{and}\quad \boldsymbol{a}(\gamma)\restriction R_s\in\prod_{i\in R_s} U_i^{1-l(s)}.$$

Thus for every $i<n$, $\{\langle a(\beta,2i), a(\beta,2i+1)\rangle, \langle a(\gamma,2i), a(\gamma,2i+1)\rangle\}$ is OP. By the method F_0 was uniformized. If $i\neq j$, then $\{\langle a(\beta,2i), a(\beta,2i+1)\rangle, \langle a(\gamma,2j), a(\gamma,2j+1)\rangle\}$ is OP. Hence $f_\beta\cup f_\gamma$ is OP, and by the uniformization its graph is cycle free, so $f_\beta\in f_\gamma\in P$. This concludes the proof of 9.1. □

Let $\mathbf{A}=\{A_i\mid i<\aleph_1\}\subseteq K$ be a family of pairwise disjoint sets such that for every $i<\aleph_1$, A_i is dense in $\bigcup_{j<\aleph_1} A_j$. Let $M(\mathbf{A})$ be a model whose universe is $A\overset{\text{def}}{=}\bigcup_{j<\aleph_1} A_j$. $M(\mathbf{A})$ has a binary relation which denotes the linear ordering which A inherits from $\mathbb{R}$, and it has unary predicates P_i which denote A_i. A model of the above form is called a K-shuffle.

Axiom BA1. Every two K-shuffles are isomorphic.

Theorem 9.2. $\mathrm{MA}+\mathrm{BA1}+2^{\aleph_0}\geqslant\aleph_\alpha$ *is consistent.*

Proof. The proof follows from the methods developed so far. We start with a universe $V_0 \models \text{GCH}$ and construct first a universe $V \supseteq V_0$ in which A1 holds and $2^{\aleph_0} \geq \aleph_\alpha$. Then step by step we isomorphize all pairs of K-shuffles. We thus have to show the following claim.

Claim (A1). *Let $M(\mathbf{A})$ and $M(\mathbf{B})$ be K-shuffles, then there is a c.c.c. forcing set $P_{\mathbf{A},\mathbf{B}}$ of power $\aleph_1$ such that $\Vdash_{P_{\mathbf{A},\mathbf{B}}} M(\mathbf{A}) \cong M(\mathbf{B})$.*

Proof. Let M be a model whose universe is $\aleph_1$ and which encodes $\mathbf{A}$ and $\mathbf{B}$. Let C be M^c-thin. Let $A = \bigcup_{i<\aleph_1} A_i$ and $B = \bigcup_{i<\aleph_1} B_i$. Let $P = \{f \in P_{\aleph_0}(A \times B) \mid f$ is an OP function, for every $i<\aleph_1$, $f(A_i) \subseteq \beta_i$ and $f^{-1}(B_i) \subseteq A_i$, G_f^C is cycle free, and if $f(a) = b$, then $E^C(a)$ and $E^C(b)$ are near$\}$. $f \leq g$ if $f \subseteq g$. It follows from the proof of 9.1 that $P_{\mathbf{A},\mathbf{B}}$ is as required. □

Question 9.3. Prove that BA$\nRightarrow$BA1.

One can ask whether BA can be strengthened to say that every two members of K are isomorphic by a differentiable OP function. This strengthening is inconsistent with ZFC.

Proposition 9.4. *There are $A, B \in K$ dense in $\mathbb{R}$, such that for every uncountable 1-1 function $f \subseteq A \times B$ there is $a \in \text{Dom}(f)$ such that every neighborhood of a contains uncountably many elements of* Dom(f) *and*

$$\lim_{\text{Dom}(f) \ni b \to a} \frac{f(b)-f(a)}{b-a} \quad \textit{does not exist.}$$

Proof. For $r \in \mathbb{Q}$ we construct $A_r, B_r \subseteq \mathbb{R}$ such that: (1) A_r, B_r are countable and contain r; (2) A_r, B_r have the order type of the rationals; and (3) for every M there is $\delta > 0$ such that for every $r, s \in \mathbb{Q}$ and for every $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle \in A_r \times B_s$ if $a_1 \neq a_2$ and $b_1 \neq b_2$ and $|a_1 - a_2|$, $|b_1 - b_2| < \varepsilon$, then

$$\left|\frac{b_2 - b_1}{a_2 - a_1}\right| > M \quad \text{or} \quad \left|\frac{a_2 - a_1}{b_2 - b_1}\right| > M.$$

The construction of such a system of sets is done unductively, and if we define $A = \bigcup_{r \in \mathbb{Q}} \text{cl}(A_r)$ and $B = \bigcup_{r \in \mathbb{Q}} \text{cl}(B_r)$, then it is easy to see that A and B are as required. □

Question 9.5. *Is it consistent that there is $A \in K$ such that every two dense subsets of A which belong to K are isomorphic by a differentiable function?*

So far we have presented several techniques for constructing forcing sets. It seems that there is a large group of consistency results concerning the structure of

K that can be proved using the methods presented. However, we did not try to find an exact formulation to the scope of consistency results that can be proved using these methods. Instead we bring several examples how the various methods can be combined to yield universes in which K has quite a diverse range of properties.

It seems to us that the techniques that have been presented so far, suffice in order to prove any consistency result about the structure of K which is consistent with MA and $2^{\aleph_0} = \aleph_2$. However, we did not make an attempt to formulate what are exactly those consistency results about K which can be proved by our models. In the sequel we prove some consistency results in which we apply the previous methods, and which hopefully exemplify the power of the above methods.

Let $A \perp B$ mean that there is no $C \in K$ such that $C \leqslant A$ and $C \leqslant B$, in this case we say that A and B are far; $A \perp\!\!\!\perp B$ denotes that $A \perp B$ and $A \perp B^*$, and this case we say that A and B are *monotonically far* (M-far).

Suppose that $G \perp\!\!\!\perp H$, and that we want to isomorphize A and B without destroying the M-farness of G and H. In the following lemma we show how this can be done.

Lemma 9.6 (A1). *Let $\lambda < 2^{\aleph_1}$; for every $i < \lambda$ let $G_i, H_i \in K$ be such that $G_i \perp H_i$. Let $A, B \in K$ be such that for every $i < \lambda$, $A \perp\!\!\!\perp G_i$ and $B \perp\!\!\!\perp G_i$. Then there is a c.c.c. forcing set P of power $\aleph_1$ such that $\Vdash_P A \cong B$, and for every $i < \lambda$, $\Vdash_P G_i \perp H_i$, if G_i is increasing, then $\Vdash_P$ "G_i is increasing", and if G_i is 2-entangled then $\Vdash_P$ "G_i is 2-entangled".*

Proof. We first construct a model which encodes all the information we need. Let $h : A \cup B \to \aleph_1$ be a 1-1 function, and for every $i < \lambda$, let $h_i : A \cup B \cup G_i \cup H_i \to \aleph_1$ be a 1-1 onto function containing h. Let M be the following model: $|M| = \aleph_1 \cup \lambda$; M has a three-place relation $R \overset{\text{def}}{=} \{\langle i, \alpha, \beta\rangle \mid h_i^{-1}(\alpha) < h_i^{-1}(\beta)\}$, we denote $\alpha <_i \beta$ to mean that $\langle i, \alpha, \beta\rangle \in R$; M has unary predicates which represent $h(A)$ and $h(B)$; and finally M has the binary relations $S_G = \{\langle i, \alpha\rangle \mid \alpha \in h_i(G_i)\}$ and $S_H = \{\langle i, \alpha\rangle \mid \alpha \in h_i(H_i)\}$.

Let $\alpha <^0 \beta$ denote that $h^{-1}(\alpha) < h^{-1}(\beta)$, hence $<^0$ is definable in M. Let C be M^c-thin, and let $P = P(C, <^0, A, B)$ be as defined in 2.1. By 9.1, P is c.c.c., and it isomorphizes A and B. We next show that for every i, $\Vdash_P G_i \perp H_i$. Suppose by contradiction $p \Vdash_P \neg(G_i \perp H_i)$. We denote $G = G_i$, $H = H_i$ and $< \, = \, <_i$. By abuse of notation we assume $A \cup B \cup G \cup H = \aleph_1$. Let τ be a P-name such that $p \Vdash_P$ "τ is an uncountable OP function and $\tau \subseteq G \times H$. W.l.o.g. $p = 0$. Let $\{\langle f_\alpha, \langle a_\alpha, b_\alpha\rangle\rangle \mid \alpha < \aleph_1\}$ be such that for every α, $f_\alpha \Vdash_P \langle a_\alpha, b_\alpha\rangle \in \tau$, and $\alpha \neq \beta \Rightarrow \langle a_\alpha, b_\alpha\rangle \neq \langle a_\beta, b_\beta\rangle$. We will reach a contradiction if we find α and β such that $f_\alpha \cup f_\beta \in P$, but $\{\langle a_\alpha, b_\alpha\rangle, \langle a_\beta, b_\beta\rangle\}$ is not OP. We uniformize $\{\langle f_\alpha, \langle a, b\rangle\rangle \mid \alpha < \aleph_1\}$ as in 9.1, hence we denote $f_\alpha = \{\langle a(\alpha, 0), a(\alpha, 1)\rangle, \ldots, \langle a(\alpha, 2n-2), \; a(\alpha, 2n-1)\rangle\}$, and we denote $a_\alpha = a(\alpha, 2n)$ and $b_\alpha = a(\alpha, 2n+1)$. W.l.o.g. all the $a(\alpha, 1)$'s are distinct. Let $\boldsymbol{a}(\alpha) = \langle a(\alpha, 0), \ldots, a(\alpha, 2n+1)\rangle$, $F_1 = \{\boldsymbol{a}(\alpha) \mid \alpha < \aleph_1\}$ and let F be the closure of F_1 in $((\aleph_1, <))^{2n+2}$. We define D, γ_0, $a(i)$, $\boldsymbol{a}$, W etc. as in 9.1. In the duplication argument we distinguish between two cases.

Case 1. $E^c(a(2n)) = E^c(a(2n+1))$. Let v be such that $E^c(a(2n)) = E^V$. We define φ_s inductively as in 9.1, except in the case when $s = v$. Suppose φ_{v+1} has been defined.

$$M^c \vDash (\forall \alpha < \aleph_1)\,(\exists \boldsymbol{x}_v)\,(\mathrm{Rng}(\boldsymbol{x}_v) > \alpha \wedge (x_{2n} \in G) \wedge (x_{2n+1} \in H) \wedge \varphi_{v+1}(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{v-1}, \boldsymbol{x}_v)).$$

Since $G \perp H$,

$$M^c \vDash (\exists \boldsymbol{x}_v^0, \boldsymbol{x}_v^1)\left(\mathrm{Rng}(x_v^0) \cap \mathrm{Rng}(\boldsymbol{x}_v^1) = \emptyset \wedge \bigwedge_{l=0}^{1} (x_{2n}^l \in G) \wedge \bigwedge_{l=0}^{1} (x_{2n+1}^l \in H) \wedge (\{\langle x_{2n}^0, x_{2n+1}^0\rangle, \langle x_{2n}^1, x_{2n+1}^1\rangle\} \text{ is OR}) \wedge \bigwedge_{l=0}^{1} \varphi_{v+1}(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{v-1}, \boldsymbol{x}_v^l)\right).$$

For every $i \in R_v$ and $l = 0, 1$, let U_i^l be an interval definable from D such that all the U_i^l's so far defined are pairwise disjoint and

$$M^c \vDash (\exists \boldsymbol{x}_v^0, \boldsymbol{x}_v^1)\left(\bigwedge_{l=0}^{1} \boldsymbol{x}_v^l \in \prod_{i \in R_v} U_i^l \wedge (\{\langle x_{2n}^0, x_{2n+1}^0\rangle, \langle x_{2n}^1, x_{2n+1}^1\rangle\} \text{ is OR}) \wedge \bigwedge_{l=0}^{1} \varphi_{v+1}(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{v-1}, \boldsymbol{x}_v^l)\right).$$

Let $\varphi_v(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{v-1})$ be the formula obtained from the above formula by substituting $\boldsymbol{a}_s$ by $\boldsymbol{x}_s$ for every $s < v$.

As in 9.1, we can find for every $s < k$, $l(s) \in \{0, 1\}$ such that for every $i < n$: if $2i \in R_s$, $2i+1 \in R_t$, then $\langle U_{2i}^{l(s)} \times U_{2i+1}^{l(t)}, U_{2i}^{1-l(s)} \times U_{2i+1}^{1-l(t)}\rangle$ is OP. We continue as in 9.1 and find β, γ such that for every $s < k$

$$\boldsymbol{a}(\beta) \restriction R_s \in \prod_{i \in R_s} U_i^{l(s)} \quad \text{and} \quad \boldsymbol{a}(\gamma) \restriction R_s \in \prod_{i \in R_s} U_i^{1-l(s)}.$$

It follows that $f_\beta \cup f_\gamma \in P$ and that $\{\langle a_\beta, b_\beta\rangle, \langle a_\gamma, b_\gamma\rangle\}$ is OR. A contradiction.

Case 2. $E^c(a(2n)) \neq E^c(a(2n+1))$. Let $E^c(a(2n)) = E^v$ and $E^c(a(2n+1)) = E^w$.

Case 2.1: E^v and E^w are not in the same component of G_f. In this case we define φ_s, $s \leq k$, exactly as in 9.1. Let S_0 be a set such that $v, w \in S_0$, and for every component L of G_f, $|S_0 \cap \{s \mid E^s \in L\}| = 1$. We define S_i as in 9.1. Next we define $l(s)$ for every $s \in S_0$. For every $s \in S_0 - \{w\}$, let $l(s) = 0$. We define $l(w)$ to be equal to 0 or 1 according to whether $\{\langle U_{2n}^0 \times U_{2n+1}^0\rangle, \langle U_{2n}^1 \times U_{2n+1}^1\rangle\}$ is OR or OP. We now define $l(s)$ for $s \in S_i$ by induction on i as in 9.1. Let $\beta, \gamma < \aleph_1$ be such that for every $s < k$,

$$\boldsymbol{a}(\beta) \restriction R_s \in \prod_{i \in R_s} U_i^{l(s)} \quad \text{and} \quad \boldsymbol{a}(\gamma) \restriction R_s \in \prod_{i \in R_s} U^{1-l(s)}.$$

It is easy to see that $f_\beta \cup f_\gamma \in P$ and that $\{\langle a_\beta, b_\beta\rangle, \langle a_\gamma, b_\gamma\rangle\}$ is OR. A contradiction.

Case 2.2: E^v and E^w are in the same component of G_f. Let $v = v_0, v_1, \ldots, v_r = w$ be such that $E^{v_0}, \ldots, E^{v_r}$ is the unique path in G_f connecting E^v

and E^w. By the symmetry between the roles of A and B we can assume that E^{v_0} and E^{v_1} are connected by $\langle a(2j), a(2j+1)\rangle$ where $a(2j)\in E^{v_0}$ and $a(2j+1)\in E^{v_1}$. We define φ_s for $s\leqslant k$ inductively. φ_k is defined as in 9.1. If $s\neq v$, then φ_s is defined from φ_{s+1} as in 9.1. Suppose φ_{v+1} has been defined, and we define φ_v.

$$M^c \models Qx_v\ (\varphi_{v+1}(\boldsymbol{a}_0,\dots,\boldsymbol{a}_{v-1},\boldsymbol{x}_v)\wedge x_{2j}\in A\wedge x_{2n}\in G).$$

Since $A \perp\!\!\!\perp G$, for $\mathrm{Z}=\mathrm{P}$ and $\mathrm{Z}=\mathrm{R}$:

$$M^c\models(\exists \boldsymbol{x}^{0,\mathrm{Z}}, \boldsymbol{x}_v^{1,\mathrm{Z}})\left(\bigwedge_{l=0}^{1}\varphi_{v+1}(\boldsymbol{a}_0,\dots,\boldsymbol{a}_{v-1},\boldsymbol{x}^{l,\mathrm{Z}})\wedge(\{\langle x_{2j}^{0,\mathrm{Z}}, x_{2n}^{0,\mathrm{Z}}\rangle, \langle x_{2j}^{1,\mathrm{Z}}, x_{2n}^{1,\mathrm{Z}}\rangle\}\text{ is OZ})\right).$$

Let $U_i^{l,\mathrm{Z}}$, $l=0,1$, $\mathrm{Z}=\mathrm{P},\mathrm{R}$ and $i\in R_v$ be pairwise disjoint open intervals disjoint from the previously defined I's definable from D such that: (1) $\langle U_{2j}^{0,\mathrm{Z}}\times U_{2n}^{0,\mathrm{Z}}, U_{2j}^{1,\mathrm{Z}}\times U_{2n}^{1,\mathrm{Z}}\rangle$ is OZ; and

$$\text{(2)}\qquad M^c\models \bigwedge_{\mathrm{Z}\in\{\mathrm{P},\mathrm{R}\}}\ \bigwedge_{l=0}\ \exists \boldsymbol{x}_v^{l,\mathrm{Z}}\left(\boldsymbol{x}_v^{l,\mathrm{Z}}\in\prod_{i\in R_v}U_i^{l,\mathrm{Z}}\wedge\varphi_{v+1}(\boldsymbol{a}_0,\dots,\boldsymbol{a}_{v-1},\boldsymbol{x}^{l,\mathrm{Z}})\right).$$

Let φ_v be the formula obtained from the above formula by replacing each $\boldsymbol{a}_s$, $s<v$, by $\boldsymbol{x}_s$. This concludes the definition of the φ_s's.

Our next goal is to define $l(s)$ for every $s<k$. In fact we have also to decide whether to duplicate the $U_i^{l,\mathrm{P}}$'s or the $U_i^{l,\mathrm{R}}$'s. Let $T=\{s\mid E^s$ and E^v are connected in $G_f\}$. We define $l(s)$ for $s\in k-T$ as in 9.1. So it remains to define $l(s)$ for $s\in T$. Let $S_0=\{w\}$, and we define S_i inductively as in 9.1. Note that $v\in S_r$. For $t<r$ we define $l(s)$ for $s\in S_t$ as in 9.1. Let Z^0 be defined as follows. If $\langle U_{2j+1}^{l(v_1)}\times U_{2n+1}^{l(w)}, U_{2j+1}^{1-l(v_1)}\times U_{2n+1}^{1-l(w)}\rangle$ is OP, then $\mathrm{Z}^0=\mathrm{R}$, and if the above pair of sets is OR, then $\mathrm{Z}^0=\mathrm{P}$. We denote each U_i^{l,Z^0} by U_i^l and proceed in the definition of $l(s)$ as in 9.1. Let β and γ be such that for every $s<k$,

$$\boldsymbol{a}(\beta)\restriction R_s\in\prod_{i\in R_s}U_i^{l(s)}\quad\text{and}\quad \boldsymbol{a}(\gamma)\restriction R_s\in\prod_{i\in R_s}U^{1-l(s)}.$$

By the proof of 9.1, $f_\beta\cup f_\gamma\in P$. We check that $\{\langle a_\beta, b_\beta\rangle,\langle a_\gamma, b_\gamma\rangle\}$ is OR. By the construction of $l(s)$, $\langle U_{2j}^{l(v)}\times U_{2j+1}^{l(v_1)}, U_{2j}^{1-l(v)}\times U_{2j+1}^{1-l(v_1)}\rangle$ is OP. $\langle U_{2j}^{l(v)}\times U_{2n}^{l(v)}, U_{2j}^{1-l(v)}\times U_{2n}^{1-l(v)}\rangle$ was chosen to be OR or OP according to whether $\langle U_{2j+1}^{l(v_1)}\times U_{2n+1}^{l(w)}, U_{2j+1}^{1-l(v_1)}\times U_{2n+1}^{1-l(w)}\rangle$ was OP or OR. Since the composition of an OP and an OR function is OR it follows that $\langle U_{2n}^{l(v)}\times U_{2n+1}^{l(w)}, U_{2n}^{1-l(w)}\times U_{2n+1}^{1-l(w)}\rangle$ is OR. Since $\langle a_\beta, b_\beta\rangle\in U_{2n}^{l(v)}\times U_{2n+1}^{l(w)}$ and $\langle a_\gamma, b_\gamma\rangle\in U_{2n}^{1-l(v)}\times U_{2n+1}^{1-l(w)}$ it follows that $\{\langle a_\beta, b_\beta\rangle,\langle a_\gamma, b_\gamma\rangle\}$ is OR. Hence we reach a contradiction again.

The proof that, if G_i is 2-entangled in V, then it remains 2-entangled after forcing with P, is very similar to the above proof. So is the proof that G_i remains an increasing set, if it was increasing in V. □

Lemma 9.7 (A1). *Let $\lambda<2^{\aleph_1}$; for every $i<\lambda$ let $G_i, H_i\in K$ be such that $G_i\perp H_i$. Suppose Q is a c.c.c. forcing set such that $\Vdash_Q\neg(G_0\perp H_0)$. Then there is a c.c.c. forcing set R of power $\aleph_1$ such that $\Vdash_R$ (Q is not c.c.c.)$\wedge(\forall i<\lambda)(G_i\perp H_i)$.*

Proof. This is an instance of the explicit contradiction method, and it is very similar to Claim 1 in 8.5. So we omit the details.

Let M be a model encoding G_i, H_i as in 9.6, and let C be M^c-thin. Let τ be a Q-name such that $\Vdash_Q$ "τ is an uncountable OP function and $\tau \subseteq G_0 \times H_0$". Let

$$\{\langle q_\alpha, a(\alpha,0,0), a(\alpha,0,1), a(\alpha,1,0), a(\alpha,1,1)\rangle \mid \alpha < \aleph_1\}$$

be a sequence with the following properties: denote

$$\boldsymbol{a}(\alpha, l) = \langle a(\alpha, l, 0), a(\alpha, l, 1)\rangle \quad \text{and} \quad \boldsymbol{a}(\alpha) = \boldsymbol{a}(\alpha, 0) \wedge \boldsymbol{a}(\alpha, 1);$$

then (1) $q_\alpha \Vdash \bigwedge_{l=0}^{1} \boldsymbol{a}(\alpha, l) \in \tau$, and (2) if $\langle \alpha, l\rangle \neq \langle \beta, m\rangle$, then

$$\{E^c(a(\alpha, l, 0)), E^c(a(\alpha, l, 1)\} \cap \{E^c(a(\beta, m, 0)), E^c(a(\beta, m, 1))\} = \emptyset.$$

Let $\alpha, \beta < \aleph_1$. We say that q_α, q_β are explicitly contradictory if there is $l(\alpha, \beta) = l \in \{0, 1\}$ such that $\{\boldsymbol{a}(\alpha, l), \boldsymbol{a}(\beta, l)\}$ is OR. Clearly if q_α and q_β are explicitly contradictory, then they are incompatible in Q. Let $R = \{\sigma \in P_{\aleph_0}(\aleph_1) \mid$ for every distinct $\alpha, \beta \in \sigma$, q_α and q_β are explicitly contradictory$\}$. $\sigma \leqslant \eta$ if $\sigma \subseteq \tau$. The proof that R is as required is very similar to the proof of claim 1 in 8.5, hence we leave it to the reader. □

In the next two lemmas we make the preparation for the use of the tail method under the assumption A1. Let $A \subsetapprox B$ denote that $|A - B| \leqslant \aleph_0$.

Lemma 9.8 (A1). *Let $\lambda < 2^{\aleph_1}$; for every $i < \lambda$ let $C_i \subseteq \aleph_1$ be a club. Then there is a club $C \subseteq \aleph_1$ such that $C \subsetapprox C_i$ for evey $i < \lambda$.*

Proof. Let $M = \langle \lambda, <, R\rangle$ where $R = \{\langle \alpha, i\rangle \mid \alpha \in C_i\}$. It is easy to see that if C is M-thin, then for every $i < \lambda$, $C \subsetapprox C_i$. □

Lemma 9.9 (A1). *Let $\lambda < 2^{\aleph_1}$; for every $i < \lambda$ let $Ai = \{a(i, \alpha) \mid \alpha < \aleph_1\} \in K$, where $\{a(i, \alpha) \mid \alpha < \aleph_1\}$ is a 1-1 enumeration of A_i, and let $A \in K$. Then there is $B \subseteq A$ and for every $i < \lambda$ a club $C_i \subseteq \aleph_1$ such that $B \in K$ and is dense in A, and if $B_i = \{a(i, \alpha) \mid \alpha \in C_i\}$, then B_i is dense in A_i, $B \perp\!\!\!\perp B_i$ and B is 2-entangled.*

Proof. Let $\{a(\lambda, \alpha) \mid \alpha < \aleph_1\}$, be a 1-1 enumeration of A; let $h : (\lambda + 1) \times \aleph_1 \to \aleph_1$ be defined as follows: $h(i, \alpha) = a(i, \alpha)$. Let $R = \{\langle i, \alpha, \beta\rangle \mid a(i, \alpha) < a(i, \beta)\}$. Let $M = \langle \lambda + 1, <, h, R\rangle$, and let C be M^c-thin. Let $\{E_\alpha \mid \alpha < \aleph_1\}$ be an enumeration of $\mathscr{E}^C$ in an increasing order. Let $D \subseteq \aleph_1$ be a club such that $|\aleph_1 - D| = \aleph_1$. For every $i < \lambda$ let $C_i = \bigcup \{E_\alpha \mid \alpha \in D\}$ and $B_i = \{a(i, \alpha) \mid \alpha \in C_i\}$. Clearly C_i is a club, $B_i \in K$ and is dense in A_i. It is easy to find $B \subseteq A$ with the following properties:

(1) if $a(\lambda, \alpha), a(\lambda, \beta) \in B$ and are distinct, then $E^C(a(\lambda, \alpha)) \neq E^C(a(\lambda, \beta))$ and $\alpha \notin \bigcup \{E_\gamma \mid \gamma \in D\}$; and

(2) $B \in K$ and is dense in A.

Suppose by contradiction $f \subseteq B \times B_i$ is an uncountable monotonic function. Let

$f' = \{\langle \alpha, \beta\rangle \mid \langle h(\lambda, \alpha), h(i, \beta)\rangle \in f\}$, $F = \text{cl}(f)$ and $F' = \{\langle \alpha, \beta\rangle \mid \langle h(\lambda, \alpha), h(i, \beta)\rangle \in F\}$. There is $d \in |M^c|$ such that F is definable from d. Let $\gamma_0 < \aleph_1$ be such that for every $\alpha \geqslant \gamma_0$, if $\alpha \in C$, then there is $N \prec M^c$ such that $d \in |N|$ and $|N| \cap \aleph_1 = \alpha$. Let $\langle \alpha, \beta\rangle \in f'$ and $\gamma_0 \leqslant \alpha, \beta$. W.l.o.g. $\alpha < \beta$. By the definition of B and C_i, there is $\gamma \in C$ such that $\alpha < \gamma \leqslant \beta$. Let $N \prec M^c$ be such that $d \in |N|$ and $|N| \cap \aleph_1 = \gamma$. Hence $\alpha \in |N| \not\ni \beta$. However, $|F'(\alpha)| \leqslant 2$ hence β is definable from α and d, and thus $\beta \in |N|$, a contradiction.

The 2-entangledness of B is proved similarly. □

Let $A \in K$. A is *into rigid* (I-rigid) if there is no monotonic $f: A \to A$ other than the identity. Let RHA be the following axiom:

Axiom RHA. For every $A \in K$ there are $B, C \subseteq A$ such that $B, C \in K$ and are dense in A, B is I-rigid and C is homogeneous.

Note that RHA $\Rightarrow \neg$CH.

Theorem 9.10. *Let* $V \vDash$ CH, *and* λ *be a regular cardinal in* V *satisfying* $\lambda^{\aleph_1} = \lambda$ *and* $\sum_{\mu<\lambda} 2^\mu = \lambda$. *Then there is a forcing set* $P \in V$ *such that* $\Vdash_P (2^{\aleph_1} = \lambda) \wedge \text{MA} \wedge \text{RHA} \wedge \text{NA}$.

Remark. The assumption that $\sum_{\mu<\lambda} 2^\mu = \lambda$ is needed just for MA and not for RHA or NA.

Proof. Let $V_0 \supseteq V$ and $V_0 \vDash (2^{\aleph_1} = \lambda) \wedge \text{A1}$. The construction, of such V_0 is done in 5.4. Let $\{\tau_i \mid i < \lambda\}$ be an enumeration of $H(\lambda)$ such that for every $\tau \in H(\lambda)$, $|\{i \mid \tau_i = \tau\}| = \lambda$. We regard each τ_i as a task of one of the following types: if τ_i is a name of an element of K, we shall find two subsets H and R of τ_i belonging to K; we shall make H homogeneous, and will make some obligations which will assure that R will be I-rigid. If τ_i is a name of a pair of members of K, then we will define π_i to make these two sets near to one another. If τ_i is a name of a c.c.c. forcing set, then either we make τ_i the next step in the interation or by the explicit contradiction method we destroy the c.c.c.-ness of τ_i.

We define a finite support iteration $\langle \{P_i \mid i \leqslant \lambda\}, \{\pi_i \mid i < \lambda\}\rangle$. Along with the construction of the π_i's we also define some obligations. An obligation which is added in the ith stage of the iteration is a P_i-name of an object of the following form $\langle H, \{d_\alpha \mid \alpha < \aleph_1\}\rangle$ where $H, \{d_\alpha \mid \alpha < \aleph_1\} \in K$, $\{d_\alpha \mid \alpha < \aleph_1\}$ is a 1-1 enumeration and $\Vdash_{P_i} (H \perp\!\!\!\perp \{d_\alpha \mid \alpha < \aleph_1\})$. If $s = \langle \hat{H}, \{\hat{d}_\alpha, \mid \alpha < \aleph_1\}\rangle$ is an obligation, then $i(s)$ denotes the stage in which it was defined; $\hat{H}(s)$ denotes $\hat{H}$; $\hat{D}(s)$ denotes the name of the set $\{\hat{d}_\alpha \in \alpha < \aleph_1\}$ and $\hat{d}(s, \alpha)$ denotes $\hat{d}_\alpha$. If s is an obligation, then for every $j \geqslant i(s)$ we will have a club $C(s, j) \subseteq \aleph_1$ such that $\Vdash_{P_j} \hat{H}(s) \perp\!\!\!\perp \{\hat{d}(s, \alpha) \mid \alpha \in C(s, j)\}$. We denote $\{d(s, \alpha) \mid \alpha \in C(s, j)\}$ by $D(s, j)$.

Suppose $\delta < \lambda$ is a limit ordinal and for every $i < \delta$, P_i has been defined. P_δ is defined automatically. Let s be an obligation with $i(s)<\delta$. By Lemma 9.8 there is a club $C\subseteq\aleph_1$ such that $C\subseteq C(s,j)$ for every $i\leqslant j<\delta$. Let $C(s,\delta)=C$.

Suppose P_i has been defined. In order to simplify the explanation we define the value of π_i in a P_i-generic extension of V_0 instead of defining π_i itself. Hence let G be P_i-generic and $W=V_0[G]$. Suppose first that $Q\overset{\text{def}}{=}v_G(\tau_i)$ is a c.c.c. forcing set. If for every obligation s, $\Vdash_Q H(s)\amalg D(s,i)$, we define $v_G(\pi_i)=Q$; for every obligation s we define $C(s,i+1)=C(s,i)$; and we do not add new obligations. Otherwise, by 9.7 there is a c.c.c. forcing set R of power $\aleph_1$ such that $\Vdash_R$ (Q is not c.c.c.)$\wedge\forall s(H(s)\amalg D(s,i))$. Let $v_G(\pi)=R$, for every s let $C(s,i+1)=C(s,i)$, and we do not add new obligations.

Suppose that $v_G(\tau_i)\in K$ and denote $v_G(\tau_i)=A$. W.l.o.g. A is dense in $\mathbb{R}$. By Lemma 9.9 there is $B\subseteq A$ and for every s a club $C(s,i+1)$ such that $B\in K$ is dense in A and is 2-entangled, and for every s, $B\amalg D(s,i+1)$. Let $H,R\subseteq B$ be disjoint dense subsets of B such that $H,R\in K$.

By repeated application of Lemma 9.6 there is a c.c.c. forcing set P of power $\aleph_1$ such that

$$\Vdash_P (H \text{ is homogeneous})\wedge(R \text{ is 2-entangled})\wedge\forall s\ (H(s)\amalg D(s,i+1)).$$

Let $v_G(\pi_i)=P$. For every two disjoint rational intervals of R, I and J, we define a new obligation $s(R,I,J)$:

$$s(R,I,J)=\langle R\cap I,\{d_\alpha\mid\alpha<\aleph_1\}\rangle$$

where $\{d_\alpha\mid\alpha<\aleph_1\}$ is a 1-1 enumeration of $R\cap J$. For every new obligation s we define $C(s,i+1)$ to be $\aleph_1$. It is easy to check that the induction hypotheses hold.

If $v_G(\tau_i)=\langle A,B\rangle$ where $A,B\in K$, then as in the previous case we find $A'\subseteq A$, $B'\subseteq B$ and for every obligation s, $C(s,i+1)$, such that $A',B'\in K$, and for every s, $A',B'\amalg D(s,i+1)$. By 9.6 there is a c.c.c. forcing set P of power $\aleph_1$ such that $\Vdash_P A'\cong B'\wedge\forall s\ (H(s)\amalg D(s,i+1))$. Let $P=v_G(\pi_i)$.

If $v_G(\tau_i)$ is none of the above, we define $v_G(\pi_i)$ to be a trivial forcing set. This concludes the definition of P_i and π.

Let $P=P_\lambda$, let G be P-generic and $W=V_0[G]$. Let $A\in K^W$. It is easy to see that A contains a homogeneous member of K which is dense in A. For some i, $A=v_{G\cap P_i}(\tau_i)$. Let $R\subseteq A$ be as defined in the ith stage of the construction. We show that R is I-rigid in W. Suppose by contradiction $f:R\to R$ is monotonic and is not the identity. For some disjoint rational intervals I and J, $f(J\cap R)\subseteq I\cap R$, and let $s=s(R,I,J)$. Let $j\geqslant i$ be such that $f\in V_0[G\cap P_j]$. But $D(s,j)\subseteq J\cap R$ and $V_0[G\cap P_j]\vDash D(s,j)\amalg I\cap R$. A contradiction, hence R is I-rigid.

The proof that $W\vDash$MA is well known. □

Question 9.11. Is RHA$+(\forall B\in K)(\exists C,D\in K)(C,D\subseteq B\wedge(C\amalg D)$ consistent?

Question 9.12 (Baumgartner). Is it consistent with ZFC that $2^{\aleph_0}>\aleph_2$ and every two $\aleph_2$-dense sets of reals ae isomorphic?

10. The structure of K and K^H when K^H is finite

In Section 6 we started the investigation of the possible structure of K and K^H under the assumption of $MA_{\aleph_1}$. In this section we continue the investigation in this direction. Our main goal is to characterize the structure of K^H under the assumption $MA_{\aleph_1} + (K^H/\cong$ is finite). In this case we obtain a full description of the possible structure of K^H, and we also obtain quite a good description of how K is built from K^H.

We did not pursue an analogous result for the general case, still we know to construct a large variety of different K^H's. There is still a shortcoming in our proof — we do not know how to enlarge $2^{\aleph_0}$ beyond $\aleph_2$.

We make an abuse of notation and denote $K^H/\cong$ by K^H. By 6.1(d), $\preccurlyeq$ induces a partial ordering on $K^H/\cong$, hence we regard $\preccurlyeq$ as a partial ordering of $K^H/\cong$. Since $(A \cong B) \Rightarrow (A^* \cong B^*)$ $*$ too can be regarded as an operation on $K^H/\cong$. Clearly $*$ is an automorphism of order 2 of $\langle K^H/\cong, \preccurlyeq\rangle$; we call such an automorphism an involution. The main theorem in this section is the following.

Theorem 10.1. *Let $\langle L, \leqslant, *\rangle$ be a finite poset with an involution. Then* $\mathrm{CON}(MA_{\aleph_1} + (\langle K^H \cup \{\emptyset\}/\cong, \preccurlyeq, *\rangle \cong \langle L, \leqslant, *\rangle))$ *iff $\langle L, \leqslant, *\rangle$ is a distributive lattice with an involution.*

We start with the easy direction of Theorem 10.1.

Definition 10.2. Let $\mathscr{A} \subseteq K^H$; $\mathscr{A}$ generates K^H if every element of K^H is a shuffle of a countable subset of $\mathscr{A}$. K^H is countably generated if there is $\mathscr{A} \subseteq K^H$ such that $|\mathscr{A}| \leqslant \aleph_0$ and $\mathscr{A}$ generates K^H.

Lemma 10.3 ($MA_{\aleph_1}$). (a) *K^H is a σ-complete upper-semilattice, that is, every countable subset of K^H has a least upper bound.*

(b) *If K^H is countably generated, then $K^H \cup \{\emptyset\}$ is a distributive complete lattice.*

(c) *If $\langle K^H, \preccurlyeq\rangle$ is well founded, then for $A \in K$ there is a nwd subset B of A such that $A - B$ is the ordered sum of members of K^H.*

Proof. (a) is just a reformulation of 6.1(f).

(b) Suppose K^H is countably generated, then by (a), K^H is a complete upper-semilattice, but then $K^H \cup \{\emptyset\}$ is also a complete lattice. Let $K^H \cup \{\emptyset\}$ be denoted by K^{HZ}. In order to show that K^{HZ} is distributive it suffices to show one of the distributive laws. We show that $(a_1 \vee a_2) \wedge b = (a_1 \wedge b) \vee (a_2 \wedge b)$. In fact we can show somewhat more: $(\bigwedge_{i \in \omega} a_i) \wedge b = \bigwedge_{i \in \omega} (a_i \wedge b)$. We do not know however whether the dual identity holds.

Let us denote the operations in K^{HZ} by $\dot\wedge$ and $\dot\vee$. Let $A \in K^{HZ}$ and for every $i \in \omega$, let $B_i \in K^{HZ}$ and suppose that $A \preccurlyeq \dot\bigvee_{i \in \omega} B_i$. We prove the following claim:

(∗) There are $A_i \in K^{HZ}$ such that $A_i \preccurlyeq B_i$ and $A = \dot\bigvee_{i \in \omega} A_i$.

W.l.o.g. each B_i is dense in $\mathbb{R}$, hence $B \stackrel{\text{def}}{=} \bigcup_{i\in\omega} B_i = \dot{\bigvee}_{i\in\omega} B_i$. Let $f: A \to B$ be OP. Recall that C^m denotes a mixing of C. If $|C| \leqslant \aleph_0$, let $C^m = \emptyset$. For every $i \in \omega$, let $C_i = f^{-1}(B_i)$ and $A_i = C_i^m$. $C_i \preccurlyeq A$, B_i hence by 6.1(g), $A_i \preccurlyeq A$, B_i. Hence $\dot{\bigvee}_{i\in\omega} A_i \preccurlyeq A$. On the other hand by picking appropriate copies of C_i^m as A_i, one can assume that $A \subseteq \bigcup_{i\in\omega} A_i$, and since $\bigcup_{i\in\omega} A_i = \dot{\bigvee}_{i\in\omega} A_i$ it follows that $A \preccurlyeq \dot{\bigvee}_{i\in\omega} A_i$. Hence $A = \dot{\bigvee}_{i\in\omega} A_i$. We have thus proved (*).

Let $A \in K^{HZ}$ and for every $i \in \omega$, $B_i \in K^{HZ}$. $A \mathbin{\dot\wedge} \dot{\bigvee}_{i\in\omega} B_i \preccurlyeq \dot{\bigvee}_{i\in\omega} B_i$, hence there are $A_j \preccurlyeq B_i$ such that $\dot{\bigvee}_{i\in\omega} A_i = A \mathbin{\dot\wedge} \dot{\bigvee}_{i\in\omega} B_i$. $A_i \preccurlyeq A$, B_i, hence $A_i \preccurlyeq A \wedge B_i$. Thus $A \mathbin{\dot\wedge} \dot{\bigvee}_{i\in\omega} B_i = \dot{\bigvee}_{i\in\omega} A_i \preccurlyeq \dot{\bigvee}_{i\in\omega}(A \mathbin{\dot\wedge} B_i)$. The inequality $\dot{\bigvee}_{i\in\omega}(A \mathbin{\dot\wedge} B_i) \preccurlyeq A \mathbin{\dot\wedge} \dot{\bigvee}_{i\in\omega} B_i$ holds in every lattice. in

(c) Suppose $\langle K^H, \preccurlyeq \rangle$ is well founded, and let $A \in K$. It suffices to show that every non-empty open interval of A contains a homogeneous subinterval. If A_1, A_2 are intervals of A and $A_1 \subseteq A_2$, then $A_1^m \preccurlyeq A_2^m$. Let A_1 be an interval of A, since K^H is well founded A_1 has a non-empty open subinterval A_2 such that for every subinterval A_3 of A_2, $A_3^m \cong A_2^m$. We show that $A_2^m \preccurlyeq A_2$. For every subinterval I of A_2, there is a family $\{g_{I,i} \mid i \in \omega\}$ of OP functions such that $\mathrm{Rng}(g_{I,i}) = I$ and $\bigcup_{i\in\omega} \mathrm{Dom}(g_{I,i}) = A_2^m$. Let $\mathcal{I}$ be a countable dense family of subintervals of A_2. Then A_2^m, A_2 and $\{g_{I,i} \mid i \in \mathcal{I}, i \in \omega\}$ satisfy the conditions of 6.1(b), hence $A_2^m \preccurlyeq A_2$. $A_2 \preccurlyeq A_2^m$, hence by 6.1(g), $A_2 \cong A_2^m$. □

We next turn to the proof of the other direction of Theorem 10.1. Let $\langle L, \vee, \wedge, * \rangle$ be a finite distributive lattice with an involution. $a \in L$ is indecomposable if for no $b, c < a$, $b \vee c = a$. Let $I(L)$ denote the set of indecomposable elements of L. Clearly $I(L)$ is closed under $*$, and every element of L is a sum of elements in $I(L)$. The following proposition shows that $I(L)$ determines L uniquely, and will guide us in the construction of a universe in which $K^{HZ} \cong L$.

Proposition 10.4. (a) *Let $\langle A, \leqslant, * \rangle$ be a finite partially ordered set with an involution. Then there is a unique distributive lattice with an involution L such that $\langle I(L), \leqslant, * \rangle \cong \langle A, \leqslant, * \rangle$.*

(b) (MA) *Let $\{A_i \mid i < n\} \subseteq K^H$ be such that: for no $j < n$, A_j is a shuffle of other members of $\{A_i \mid i < n\}$, and for every $A \in K^H$, A is a shuffle of some members of $\{A_i \mid i < n\}$. Then K^{HZ} is finite, and $I(K^{HZ}) = \{A_i \mid i < n\}$.*

Proof. Easy. □

By the above proposition it is clear what has to be done in order to construct a universe in which $K^{HZ} \cong L$. We start with a universe V satisfying CH, and with a family $\{A_a \mid a \in I(L)\} \subseteq K^H$ such that no A_a is a shuffle of other A_a's and such that $a \mapsto A_a$ is an isomorphism between $\langle I(L), \leqslant, * \rangle$ and $\langle \{A_a \mid a \in I(L)\}, \preccurlyeq, * \rangle$. We then construct $W \supseteq V$ which satisfies MA, and in which every element of K^H is a shuffle of some members of $\{A_a \mid a \in I(L)\}$, and no A_a is a shuffle of other A_a's. In such a universe $K^{HZ} \cong L$.

For the rest of this section L is a fixed finite distributive lattice with an involution. W.l.o.g. $I(L)=\{0,\ldots,n-1\}$, and we denote the partial ordering on $I(L)$ by $\leqslant$. Recall that $*$ is an involution of $\langle\{0,\ldots,n-1\},\leqslant\rangle$.

The method of proof of the following lemma is well known and is due to Sierpinski [8]. We take the liberty to present a proof here since the technical details are not completely obvious.

Lemma 10.5 (CH). *There are* $\{A_i \mid i<n\}\subseteq K^H$ *such that:*

(1) *If* $i\leqslant j$, *then* $A_i\subseteq A_j$.

(2) *If* $i=j^*$, *then* $A_i=A_j^*$.

(3) *Let* $B_i=A_i-\bigcup_{j<i}A_j$. *Then, if* $i\leqslant j$, *then* $B_i\perp A_j$.

Question. Does the above lemma follow from A1?

Proof. Let $f\subseteq\mathbb{R}\times\mathbb{R}$. f is a maximal OP function if f is an OP function and there is no OP function g such that $f\subsetneqq g\subseteq\mathbb{R}\times\mathbb{R}$. Let $\{f_\alpha \mid \alpha<\aleph_1\}$ be an enumeration of all maximal OP functions. Let $\bar{g}$ denote the function such that for every z, $\bar{g}(z)=-z$; for every f let $f^*=\bar{g}\circ f\circ\bar{g}^{-1}$; and if F is a set of functions, let $F^*=\{f^* \mid f\in F\}$.

We define by induction on $\alpha<\aleph_1$ a family of pairwise disjoint countable dense subsets of $\mathbb{R}$, $\{B(i,\alpha) \mid i<n\}$, and families of OP functions $\{F(i,\alpha) \mid i<n\}$. Let $A(i,\alpha)=\bigcup_{j\leqslant i}B(j,\alpha)$. Our induction hypotheses are: (1) if $i=j^*$, then $B(i,\alpha)=B(j,\alpha)^*$ and $F(i,\alpha)=F(j,\alpha)^*$; and (2) for every $i<n$: if $f\in F(i,\alpha)$, then $f:\mathbb{R}\to\mathbb{R}$ and $f(A(i,\alpha))=A(i,\alpha)$, and for every $x,y\in A(i,\alpha)$ there is $f\in F(i,\alpha)$ such that $f(x)=y$.

It is easy to define $\{B(i,0) \mid i<n\}$ and $\{F(i,0) \mid i<n\}$. If δ is a limit ordinal, let $B(i,\delta)=\bigcup_{\alpha<\delta}B(i,\alpha)$ and $F(i,\delta)=\bigcup_{\alpha<\delta}F(i,\alpha)$.

Suppose $\{B(i,\alpha) \mid i<n\}$, $\{F(i,\alpha) \mid i<n\}$ have been defined, and we wish to define $\{B(i,\alpha+1) \mid i<n\}$. Let $\bar{B}(i,\alpha)=B(i,\alpha)\cup\bigcup\{f_\beta(B(i,\alpha)) \mid \beta<\alpha\}$. Let $U\subseteq\{0,\ldots,n-1\}$ be such that for every $i<n$, $|U\cap\{i,i^*\}|=1$. For $x\in\mathbb{R}$ and a set of functions F, let $\mathrm{cl}(x,F)$ denote the closure of x under $F\cup\{f^{-1} \mid f\in F\}$. It is easy to construct a set $\{x_i \mid i\in U\}$ such that: (1) for every $i\in U$, $\mathrm{cl}(x_i,F(i,\alpha))\cap\mathrm{cl}(-x_i,F(i^*,\alpha))=\emptyset$; and (2) for every $i\in U$, $\mathrm{cl}(x_i,F(i,\alpha))\cap\bigcup_{j<n}\bar{B}(j,\alpha)=\emptyset$.

Let $i<n$; if $i\in U$ and $i\neq i^*$ let $B(i,\alpha+1)=B(i,\alpha)\cup\mathrm{cl}(x_i,F(i,\alpha))$; if $i=i^*$ let $B(i,\alpha+1)=B(i,\alpha)\cup\mathrm{cl}(x_i,F(i,\alpha))\cup\mathrm{cl}(-x_i,F(i,\alpha))$; and if $x_i\notin U$ let $B(i,\alpha+1)=B(i,\alpha)\cup\mathrm{cl}(-x_{i^*},F(i,\alpha))$.

Since for every i, $F(i,\alpha)^*=F(i^*,\alpha)$, it follows that $B(i,\alpha+1)^*=B(i^*,\alpha+1)$. By the choice of the x_i's, $B(i,\alpha+1)\cap B(j,\alpha+1)=\emptyset$ whenever $i\neq j$. It is easy to define for every $i<n$, $F(i,\alpha+1)\supseteq F(i,\alpha)$ so that the induction hypotheses will hold.

Let $A_i=\bigcup_{\alpha<\aleph_1}A(i,\alpha)$. It is easy to see that each A_i belongs to K^H, and (1) and (2) of 10.5 hold. Suppose by contradiction $i\leqslant j$ but $N(B_i,A_j)$. Hence for some $k\neq i$, $N(B_i,B_k)$. Let f be a maximal OP function such that $|f\cap B_i\times B_k|=\aleph_1$. For some α, $f=f_\alpha$. By the construction, for every $\beta>\alpha$, $f_\alpha(B(i,\beta))\cap(B(k,\beta)-\bigcup\{B(k,\gamma)\mid\gamma<\beta\})=\emptyset$, hence $|f_\alpha(B_i)\cap B_k|\leq\aleph_0$, a contradiction. $\square$

Let $\{A_i,\mid i<n\}$ and $\{B_i\mid i<n\}$ be as assured by Lemma 10.5. For every $i<n$ and every rational interval I let $\{b(\alpha,i,I)\mid\alpha<\aleph_1\}$ be a 1-1 enumeration of $B_i\cap I$. Let F_i be the following filter: $F_i=\{B\subseteq B_i\mid$ for every rational interval I, $\{\alpha\mid b(\alpha,i,i)\in B\}$ contains a club$\}$. For simplicity we assume that if $i=j^*$, then for every I and α, $b(\alpha,i,I)=-b(\alpha,j,I^*)$; this ascertains that $\bar{g}$ is an isomorphism between F_i and F_j. Note that if $B\in F_i$, then $B\in K$ and B is dense in B_i.

For subsets of $\mathbb{R}$, $C_0,\ldots,C_{k-1}$, let $\bigwedge_{i=0}^{k-1}C_i=0$ denote the following fact: there is no $C\in K$ such that $C\preccurlyeq C_i$ for every $i<k$. Note that for $C_i\in K$, there is no meaning to $\bigwedge_{i=0}^{k-1}C_i$, since there are not meets in K.

Let $\{x_i\mid i<n\}$, $\{y_\tau\mid\tau\subseteq n\}$ be sets of variables. Let

$$\varphi_0\equiv\bigwedge_{i\neq j}(x_i\wedge x_j=0)\wedge\bigwedge_{i\neq j^*}(x_i\wedge x_j^*=0).$$

Let z^ε denote z if $\varepsilon=0$, and z^* if $\varepsilon=1$. A farness formula (F-formula) is a formula of the form $\bigwedge_{i\in I}z_i^{\varepsilon(i)}=0$ where $\{z_i\mid i\in I\}$ is any set of variables and $\varepsilon(i)\in\{0,1\}$. Let χ be an F-formula with variables belonging to $\{x_i\mid i<n\}\cup\{y_\tau\mid\tau\subseteq n\}$. We say that φ_0 implies χ ($\varphi_0\Rightarrow\chi$) if for every distributive lattice with an involution L and for every assignment s such that for every τ, $s(y_\tau)=\bigvee\{s(x_i)\mid i\in\tau\}$: if $L\models\varphi(s)$, then $L\models\chi[s]$. More explicitly,

$$\varphi_0\Rightarrow\bigwedge_{i\in I}x^{\varepsilon(i)}\wedge\bigwedge_{\tau\in J}y_\tau^{\varepsilon(\tau)}=0,$$

if there is $i_0<n$ such that for every $i\in I$, $i_0=i^{\varepsilon(i)}$, and for every $\tau\in J$, $i_0\in\{i^{\varepsilon(\tau)}\mid i\in\tau\}$.

Let $\varphi_1=\{\chi\mid\chi$ is an F-formula and $\varphi_0\Rightarrow\chi\}$. Let $C_0,\ldots,C_{n-1}\in K$ and suppose $K\models\varphi_0[C_0,\ldots,C_{n-1}]$. It is obvious that $K\models\varphi_1[s]$ where s is the assignment which maps each x_i to C_i and each y_τ to $\bigcup_{i\in\tau}C_i$. Let s be an assignment such that $\mathrm{Dom}(s)\subseteq\{x_i\mid i<n\}\cup\{y_\tau\mid\tau\subseteq n\}$. We say that $\varphi_1[s]$ holds ($\models\varphi_1[s]$), if $\chi[s]$ holds for every conjunct χ of φ_1 whose variables belong to $\mathrm{Dom}(s)$.

For every $i<n$, let $\tau_i=\{j\mid j\leqslant i\}$; let s be the assignment which maps each i to B_i and each τ_i to A_i; by the above discussion $K\models\varphi_1[s]$.

We now outline in more detail the proof of the second half of Theorem 10.1. We start with a universe V satisfying CH, and with $\{A_i\mid i<n\}$, $\{B_i\mid i<n\}$, $\langle F_i\mid i<n\rangle$ as described above. We define by induction on $\nu<\aleph_2$ a finite support iteration of c.c.c. forcing sets $\langle\{P_\nu\mid\nu\leqslant\aleph_2\}$, $\{\pi_\nu\mid\nu<\aleph_2\}$, and a sequence $\{\langle\hat{B}(\nu,0),\ldots,\hat{B}(\nu,n-1)\rangle\mid\nu>\aleph_2\}$ such that for every ν and i, $\hat{R}(\nu,i)$ is a P_ν-name, and

$$\Vdash_{P_\nu}(\forall i<n)\,(\hat{B}(\nu,i)\in F_i\wedge\hat{B}(\nu,i)^*=\hat{B}(\nu,i^*)\wedge(K\models\varphi_1[\hat{B}(\nu,0),\ldots,\hat{B}(\nu,n-1),A_0,\ldots,A_{n-1}])$$

where each $\hat{B}(\nu,i)$ replaces x_i, and each A_i is replacing y_{τ_i}.

We prepare in advance a list of tasks. There are two kinds of tasks: the first one is designed in order to take care that $\Vdash_{P_{\aleph_2}}$ MA, and the second is to assure that $\Vdash_{P_{\aleph_2}} K^H/\cong$ is finite.

If ν is a limit ordinal, then P_ν is defined automatically and $\{\hat{B}(\nu, i) \mid i<n\}$ is defined as in 4.1 or 9.10.

Suppose P_ν, $\{\hat{B}(\nu, i) \mid i<n\}$ have been defined, and we wish to define π_ν and $\{\hat{B}(\nu+1, i) \mid i<n\}$.

Case 1: Suppose the ν's task is a P_ν-name π such that $\Vdash_{P_\nu}$ "π is c.c.c.". If $\Vdash_{P_\nu}(\Vdash_\pi K \models \varphi_1[\hat{B}(\nu, 0), \ldots, \hat{B}(\nu, n-1), A_0, \ldots, A_{n-1}])$, then we define $\pi_{\nu+1}$ to be π and $B(\nu+1, i)$ to be $B(\nu, i)$. Otherwise we define $\pi_{\nu+1}$ to be the P_ν-name of the forcing set R which is constructed in the following lemma.

Lemma 10.6 (CH). *Let s be an assignment such that $K \models \varphi_1[s]$. Suppose Q is a c.c.c. forcing set such that $\Vdash_Q (K \models \neg\varphi_1[s])$. Then there is a c.c.c. forcing set R of power $\aleph_1$ such that $\Vdash_R (K \models \varphi_1[s]) \wedge (Q$ is not c.c.c$)$.*

Proof. R is constructed by the method of explicit contradiction. The details of the proof are similar to claim 1 in Lemma 8.5. Thus we give the definition of R but omit the proof that R satisfies the requirements of the lemma.

Since $\Vdash_Q \neg\varphi_1[s]$, there are $C_0, \ldots, C_{k-1} \in K^{(V)}$ such that in V, $K \models \bigwedge_{i<k} C_i = 0$, but Q forces that $K \models \neg(\bigwedge_{i<k} C_i = 0)$. Let M be a model which encodes all the relevant information. Recall that a set of two k-tuples $\{\langle a_0, \ldots, a_{k-1}\rangle, \langle b_0, \ldots, b_{k-1}\rangle\}$ is called OP, if for every $i<j<k: a_i<b_i$ iff $a_j<b_j$. Let τ be a Q-name such that $\Vdash_Q (\tau \subseteq \prod_{i<k} C_i) \wedge (|\tau| = \aleph_1) \wedge$ (every two element subset of τ is OP). The number of variables in φ_1 is $n+2^n$, accordingly let $m = 2(n+2^n)+1$. Let $\{\langle q_\alpha, \boldsymbol{a}(\alpha, 0), \ldots, \boldsymbol{a}(\alpha, m-1)\rangle \mid \alpha < \aleph_1\}$ be such that: (1) for every $\alpha < \aleph_1$ and $i<m$, $q_\alpha \Vdash_Q \boldsymbol{a}(\alpha, i) \in \tau$; (2) for every $\alpha < \aleph_1$ and $i<j<m$, there is $\gamma \in C_M$ such that $\boldsymbol{a}(\alpha, i) < \gamma \leqslant \boldsymbol{a}(\alpha, j)$; and (3) for every $\alpha < \beta < \aleph_1$, there is $\gamma \in C_M$ such that $\boldsymbol{a}(\alpha, m-1) < \gamma \leqslant \boldsymbol{a}(\beta, 0)$.

We say that p_α and p_β are explicitly contradictory if for some $i<m$, $\{\boldsymbol{a}(\alpha, i), \boldsymbol{a}(\beta, i)\}$ is not OP. Let $R = \{\sigma \in P_{\aleph_0}(\aleph_1) \mid$ for every distinct $\alpha, \beta \in \sigma$, p_α and p_β are explicitly contradictory$\}$. $\sigma_1 \leqslant \sigma_2$ if $\sigma_1 \subseteq \sigma_2$. As in 8.5 it can be proved that R satisfies the requirements of the lemma. □

Case 2: Suppose ν's task is a P_ν-name of a member B of K. In this case we define π_ν and $\hat{B}(\nu+1, i)$ according to the following lemma.

Lemma 10.7 (CH). *Let A_i, B_i, F_i be as above, let $B(\nu, i) \in F_i$ be such that $K \models \varphi_1[B(\nu, 0), \ldots, B(\nu, n-1), A_0, \ldots, A_{n-1}]$, and let $B \in K$. Then there is an interval A of B, $\tau \subseteq n$, $B_i' \in F_i$, $i<n$, and a c.c.c. forcing set P of power $\aleph_1$ such that for every $i<n$, $B_i'^* = B_{i^*}'$ and*

$$\Vdash_P (K \models \varphi_1[B_0', \ldots, B_{n-1}', A_0, \ldots, A_{n-1}]) \wedge \left(A \cong \bigcup_{i \in \tau} A_i\right).$$

The remainder of this section is devoted to the proof of the above lemma. But first we show how to define $\pi_{\nu+1}$ and $B(\nu+1, i)$, and how Theorem 10.1 follows from what has been described so far. Let $\pi_{\nu+1}$ be the P_ν-name of the forcing set P of Lemma 10.7 and $\hat{B}(\nu+1, i)$ be the P_ν-name of B'_i of Lemma 10.7.

Let $P = P_{\aleph_2}$. Clearly $\Vdash_P \mathrm{MA}$, and $\Vdash_P (\forall A \in K^H)(\exists \tau \subseteq n)(A = \bigvee_{i\in\tau} A_i)$. It remains to show that if $i \neq j$, then $\Vdash_P A_i \not\cong A_j$. Suppose the contrary. Let G be P-generic and $W = V[G]$, and let $f: A_i \to A_j$ be an isomorphism between A_i and A_j belonging to W. For some $\nu < \aleph_2$, $f \in V[G \cap P_\nu] \stackrel{\text{def}}{=} W'$. W.l.o.g. $i \not\leqslant j$. Let $B(\nu, i) = v_{G\cap P_\nu}(\hat{B}(\nu, i))$. Hence in W', $B(\nu, i) \perp A_j$, but $f(B(\nu, i)) \subseteq A_j$, a contradiction.

The proof of Theorem 10.1 will be concluded if we prove Lemma 10.7.

Let $\{x_i^l \mid i < n,\ l \in \{0, 1\}\}$, $\{y_\tau^l \mid \tau \subseteq n, l \in \{0, 1\}\}$ be sets of variables, x_i^0, x_i^1 are called copies of x_i, and y_τ^0, y_τ^1 are called copies of y_τ. A formula φ' is called a copy of φ_0, if it is gotten from φ_0 by replacing every occurance of a variable in φ_0 by one of its copies. Note that two occurances of the same variable need not be replaced by the same copy of that variable. A copy of φ_1 is defined similarly. Let ψ_0 be the conjunction of all copies of φ_0, and ψ_1 be the conjunction of all copies of φ_1. We again make the convention that for an assignment s with

$$\mathrm{Dom}(s) \subseteq \{x_i^l \mid i < n, l \in \{0, 1\}\} \cup \{y_\tau^l \mid \tau \subseteq n, l \in \{0, 1\}\},$$

$K \models \psi[s]$ means that all conjuncts of ψ_t whose variables belong to $\mathrm{Dom}(s)$ are satisfied.

Proof of 10.7. Let A_i, B_i, F_i, $B(\nu, i)$ and B be as in 10.7. We denote $B(\nu, i) = B(i)$, and let $F(i)$ be the restriction of F_i to $p(B(i))$. Note that $F(i)$ is defined from some enumerations of $B(i) \cap I$ in the same way that F_i was defined. Hence for the rest of the section we ignore B_i and F_i, and have to remember just the properties of $B(i)$ and $F(i)$.

For further reference let us recall the properties of A_i, $B(i)$ and $F(i)$. (1) A_i, $B(i)$ are dense in $\mathbb{R}$, $B(i) \in K$; (2) $\{A_i \mid i < n\} \subseteq K^H$; (3) $B(i) \subseteq A_i - \bigcup_{j<i} A_j$; (4) $\bar{g}(B(i)) = B(i^*)$; (5) for every rational interval I there is a 1-1 enumeration $\{b(\alpha, iI) \mid \alpha < \aleph_1\}$ of $B(i) \cap I$ such that $F(i) = \{B' \subseteq B(i)\} \mid$ for every rational interval I $\{\alpha \mid b(\alpha, i, I) \in B'\}$ contans a club$\}$. $b(\alpha, i^*, I^*) = -b(\alpha, i, I)$; (5) for every $i < n$ let $\tau_i = \{j \mid j \leqslant i\}$ and $A_{\tau_i} = A_i$; then $K \models \varphi_1[B(0), \ldots, B(n-1), A_{\tau_0}, \ldots, A_{\tau_{n-1}}]$.

For $D_0, \ldots, D_{k-1} \in K$, let $D \leqslant \bigwedge_{i<k} D_i$ mean that $(\forall i < k)(D \leqslant D_i)$. Let $D \in K$. We define

$$\eta(D) = \{\sigma \subseteq n \mid (\exists D' \in K)((D' \leqslant D \wedge \textstyle\bigwedge_{i\in\sigma} A_i) \wedge (\forall i \notin \sigma)(D' \wedge A_i = 0))\}.$$

Note that if $D_1 \subseteq D_2$, then $\eta(D_1) \subseteq \eta(D_2)$. Hence there is an interval A of B such that for every interval A' of A $\eta(A') = \eta(A)$.

10.7 follows from the following three lemmas.

Lemma 10.8 (CH). *Let A_i, $F(i)$, $B(i)$ be as above, and let $A \in K$ be such that for* *every interval A' of A, $\eta(A')=\eta(A)$. Then there are $\emptyset\neq\tau\subseteq n$, $\{B_1(i)\mid i\in\tau\}$ and $\{B_0(i)\mid i<n\}$ such that:*

(1) *If $j<i\in\tau$, then $j\in\tau$.*

(2) *$B_1(i)\in K$, $B_1(i)$ is a dense subset of A, and for every $i\neq j$, $B_1(i)\cap B_1(j)=\emptyset$.*

(3) *$B_0(i)\in F(i)$.*

(4) *Let $\tau_i=\{j\mid j\leqslant i\}$, $A^0_{\tau_i}=A_i$, $A^1_\tau=A$, $B^0_i=B_0(i)$ and $B^1_i=B_1(i)$. Then $K\vDash \psi_1[B^0_0,\dots,B^0_{n-1};B^1_j\mid j\in\tau;A^0_{\tau_1},\dots,A^0_{\tau_{n-1}};A^1_\tau]$.*

Lemma 10.9 (CH). *For $l=0,1$ let $n_l\leqslant n$, $B_l(0),\dots,B_l(n_l-1)\in K$, $\Gamma_l\subseteq P(n_l)$ and $\{A^l_\tau\mid\tau\in\Gamma_l\}\subseteq K$. Suppose the following conditions hold:*

(1) *$B_l(i)$ and A^l_τ are dense in $\mathbb{R}$.*

(2) *If $i\in\tau\in\Gamma_l$, then $b_l(i)\subseteq A^l_\tau$; and if $i<j<n_l$, then $B_l(i)\cap B_l(j)=\emptyset$.*

(3) *If $\tau_1,\tau_2\in\Gamma_l$ and $\tau_1\subseteq\tau_2$, then $A^l_{\tau_1}\subseteq A^l_{\tau_2}$.*

(4) *$K\vDash\psi_1[B_l(i)\mid l<2,\ i<n_l;A^l_\tau\mid l<2,\tau\in\Gamma_l]$.*

Let $t\in\{0,1\}$. Then there are pairwise disjoint $\{B^t_i\mid i<n_t\}$ such that

(1) *$K\ni B^t_i\supseteq B_t(i)$.*

(2) *For every $\tau\in\Gamma_t$, $A^t_\tau\subseteq\bigcup_{i\in\tau}B^t_i$.*

(3) *$K\vDash\psi_1[B^t_i\mid i<n_t;B_{1-t}(i)\mid i<n_{1-t};A^{1-t}_\tau\mid\tau\in\Gamma_{1-t}]$.*

Lemma 10.10 (CH). *Let $n_1\leqslant n_0\leqslant n$, let $\{B^l_i\mid l<2,i<n_l\}$, $\{D^l_i\mid l<2,i<n_1\}$ be such that all the B^l_i's and D^l_i's belong to K and are dense in $\mathbb{R}$, for every $l<2$ and $i<j<n_l$, $B^l_i\cap B^l_j=\emptyset$, $D^l_i\subseteq B^l_i$ and $K\vDash\psi_0[B^0_0,\dots,B^0_{n_0-1},B^1_0,\dots,B^1_{n_1-1}]$. Then there is a c.c.c. forcing set P of power $\aleph_1$ such that*

$$\Vdash_P\left(\bigcup_{i<n_1}D^0_i\cong\bigcup_{i<n_1}D^1_i\right)\wedge(K\vDash\psi_0[B^0_0,\dots,B^0_{n_0-1},B^1_0,\dots,B^1_{n_1-1}]).$$

Remark. Lemma 10.8 and 10.10 can be proved assuming A1; we do not know how to prove 10.9 without assuming CH; this is the reason why in 10.1 we cannot enlarge $2^{\aleph_1}$ beyond $\aleph_2$.

We first conclude the proof of 10.7 assuming 10.8–10.10. Let A be an interval of B such that for every interval A' of A, $\eta(A')=\eta(A)$. From 10.8 we obtain $\tau\subseteq n$ and $B_l(i)$'s. By renaming $\{0,\dots,n-1\}$ we can assume that $\tau=\{0,\dots,n_1-1\}$. Let us denote $\tau_i=\{j\mid j\leqslant i\}$, $A^0_{\tau_i}=A_i$, $A^1_\tau=A$, $\Gamma_0=\{\tau_0,\dots,\tau_{n-1}\}$, $\Gamma_1=\{\tau\}$, $n_0=n$ and $t=1$. (n_1 has already been defined.) The conditions of 10.9 are satisfied by the $B_l(i)$'s, A^l's etc., hence from 10.9 we obtain $\{B^1_i\mid i\in\tau\}$. By intersecting each B^1_i with A^1_τ we can assume that $\bigcup_{i\in\tau}B^1_i=A^1_\tau$, the other properties of the B^1_i's are not destroyed. Obviously

$$K\vDash\psi_1[B_0(i)\mid i<n_0;B^1_i\mid i<n;A^0_{\tau'}\mid\tau'\in\Gamma_0;A^1_\tau].$$

We can now apply Lemma 10.9 with $t=0$ to the $B_0(i)$'s, B^1_i's, $A^0_{\tau_i}$'s and A^1_τ. We thus obtain from 10.9 the B^0_i's. From 10.9 we know that $K\vDash\psi_0[B^0_0,\dots,B^0_{n-1},B^1_0,\dots,B^1_{n_1-1}]$. For $i\in\tau$ let $D^1_i=B^1_i$ and $D^0_i=B^0_i\cap\bigcup_{j\in\tau}A_j$.

The conditions of Lemma 10.10 hold, hence let P be the forcing set obtained in 10.10. Let $j \in \tau$; then $\bigcup_{i \in \tau_j} B_i^0 \supseteq A_{\tau_j} = A_j$. Since $i < j \in \tau \Rightarrow i \in \tau$, $\bigcup_{j \in \tau} \tau_j = \tau$, hence $\bigcup_{j \in \tau} D_j^0 = \bigcup_{j \in \tau} A_j$. Recall that $\bigcup_{j \in \tau} D_j^1 = A$, hence $\Vdash_P A \cong \bigcup_{j \in \tau} A_j$. $\Vdash_P (K \vdash \varphi_0[B_0^0, \ldots, B_{n-1}^0])$; for $\sigma \subseteq n$ let $A'_\sigma = \bigcup_{j \in \sigma} B_j^0$, hence for every i, $A'_{\tau_i} \supseteq A_{\tau_i}$. Clearly $\Vdash_P (K \vdash \varphi_1[B_i^0 \mid i < n; A'_{\tau_i} \mid i < n])$. Recall that for every i, $B_i^0 \supseteq B_0(i)$, hence $\Vdash_P (K \vdash \varphi_1[B_0(i) \mid i < n; A_{\tau_i} \mid i < n])$. But according to 10.8, $B_0(i) \in F(i) \subseteq F_i$. For every $i < n$, let $B'_i = B_0(i) \cap \bar{g}(B_0(i^*))$, hence $B'_i \in F_i$, $B_i'^* = B'_{i*}$, $\Vdash_P A \cong \bigcup_{j \in \tau} A_j$, and $\Vdash_P (K \vDash \varphi_1[B'_0, \ldots, B'_{n-1}, A_0, \ldots, A_{n-1}]$. This concludes the proof of 10.7. □

Proof of 10.8. W.l.o.g. A is dense in $\mathbb{R}$. Let $\tau = \{j \mid (\exists \sigma \in \eta(A))\, (j \in \bigcap_{i \in \sigma} \tau_i)\}$. Using the fact that $K \vDash \varphi_1[A_{\tau_0}, \ldots, A_{\tau_{n-1}}]$, it is easy to see that for every $\sigma \in \eta(A)$, $\bigcap_{i \in \sigma} \tau_i \neq \emptyset$ (we denote $\bigcap_{i \in \emptyset} \tau_i = n$). Since obviously $\eta(A) \neq \emptyset$ it follows that $\tau \neq \emptyset$. It is obvious that if $j < i \in \tau$, then $j \in \tau$. For every $\sigma \in \eta(A)$, let $D_\sigma \in K$ be a dense subset of $\mathbb{R}$ exemplifying that fact that $\sigma \in \eta(A)$. For every $j \in \tau$, let $D_j^1 = \bigcup \{D_\sigma \mid \sigma \in \eta \text{ and } j \in \bigcap_{i \in \sigma} \tau_i\}$. By an argument similar to Lemma 9.9, it is easy to find $\{B_0(i) \mid i < n\}$ and $B_1(i) \mid i \in \tau\}$ such that for every $i \in \tau$, $B_1(i) \in K$ and $B_1(i)$ is a dense subset of D_i^1; if $i \neq j$, then $B_1(i) \cap B_1(j) = \emptyset$, $B_0(i) \subseteq B(i)$, $B_0(i) \in F(i)$ and for every $i \neq j$

$$B_1(i) \wedge (B_1(i)^* \cup B_1(j) \cap B_1(j)^* \cup B_0(j) \cup B_0(j)^*) = 0.$$

(Here we assume that $B_1(j) = \emptyset$ if $j \notin \tau$.) Clearly (1)–(3) of 10.8 hold. Recall that A_i is denoted by $A_{\tau_i}^0$ and that $A_\tau^1 = A$. Let s be the assignment such that $s(x_i^l) = B_l(i)$ and $s(y_\sigma^l) = A_\sigma^l$. Recall that an F-formula is a formula of the form $\bigwedge_{i \in I} z_i = 0$. Let χ be an F-formula, let χ^* be the formula obtained from χ be replacing every variable z of χ by z^*. Let χ^+ be the formula obtained from χ by replacing every occurance of $(x_i^0)^*$ or $(y_\sigma^0)^*$ in χ by $x_{i^*}^0$ and $y_{\sigma^*}^0$ respectively where $\sigma^* = \{i^* \mid i \in \sigma\}$. Clearly χ is a conjunct of ψ_1 iff χ^* is, and the same holds for χ^+. Also $K \vDash \chi[s]$ iff $K \vDash \chi^*[s]$ iff $K \vDash \chi^+[s]$. Let χ be an F-formula and suppose $K \vDash \neg\chi[s]$. We show that χ is not a conjunct of ψ_1. By the definition of the $B_l(i)$'s it is clear that there is at most one occurance of a variable of the form x_i^l in χ. Replacing, if necessary, χ by χ^+, χ^* or χ^{*+} it can be assumed that

$$\chi \equiv \bigwedge_{i \in \sigma} y_{\tau_i}^0 \wedge \bigwedge \{t \mid t \in T\} = 0,$$

where T is a subset of $\{x_j^0 \mid j < n\} \cup \{x_j^1 \mid j \in \tau\} \cup \{(x_j^1)^* \mid j \in \tau\} \cup \{y_\tau^1, (y_\tau^1)^*\}$, and T intersects the union of the first three sets in at most one element. The case $T = \{x_j^0\}$ follows trivially from the fact that $B_0(j) \subseteq B(j)$ and from property (6) in 10.7. Suppose $T = \{x_j^1\}$, hence $\bigwedge_{i \in \sigma} A_{\tau_i}^0 \wedge B_1(j) \neq 0$. By the definition of $B_1(j)$ there is $\sigma' \in \eta(A)$ such that $j \in \bigcap_{i \in \sigma'} \tau_i$, and $\bigwedge_{i \in \sigma} A_{\tau_i}^0 \wedge D_{\sigma'} \neq 0$. It follows from the definition of the D_σ's that $\sigma \subseteq \sigma'$, and hence $j \in \bigcap_{i \in \sigma} \tau_i$. This means that χ is not a conjunct of ψ_1.

We next check that if $(B_j^1)^* \wedge \bigwedge_{i \in \sigma} A_{\tau_i}^0 \neq 0$, then $j^* \in \bigcap_{i \in \sigma} \tau_i$. If the above holds, then for some $\sigma' \in \eta(A)$, $j \in \bigcap_{i\in\sigma'} \tau_i$ and $(D_{\sigma'})^* \wedge \bigwedge_{i\in\sigma} A^0_{\tau_i} \neq 0$. $(A^0_{\tau_i})^* = A^0_{\rho_{i*}}$, hence $D_{\sigma'} \wedge \bigwedge_{i\in\sigma} A^0_{\tau_{i*}} \neq 0$. Let $\sigma^* = \{i^* \mid i \in \sigma\}$, hence $D_{\sigma'} \wedge \bigwedge_{i\in\sigma^*} A^0_{\tau_i} \neq 0$. By the definition of $D_{\sigma'}$, $\sigma^* \subseteq \sigma'$; hence $j \in \bigcap_{i\in\sigma^*} \tau_i$, and hence

$$j^* \in \Big(\bigcap_{i\in\sigma^*} \tau_i\Big)^* = \bigcap_{i\in\sigma^*} \tau_i^* = \bigcap_{i\in\sigma^*} \tau_{i^*} = \bigcap_{i\in\sigma} \tau_i.$$

We next check that if $(B^1_j)^* \wedge A^1_\tau \neq 0$, then $j^* \in \tau$. Suppose the above happens, and let $\sigma \in \eta(A)$ be such that $j \in \bigcap_{i\in\sigma} \tau_i$ and $(D_\sigma)^* \wedge A^1_\tau \neq 0$. Let $K \ni D \leqslant D^*_0 \wedge A^1_\tau$. Since $D_\sigma \wedge A_i = 0$ for every $i \notin \sigma$, and since $D^* \leqslant D_\sigma$, $D^* \wedge A_i = 0$ for every $i \notin \sigma$, and hence $D \wedge A_i = 0$ for every $i \notin \sigma^*$. Obviously $D^* \leqslant \bigwedge_{i\in\sigma^*} A_i$. Hence $\sigma^* \in \eta(A)$. Since $j^* \in \bigcap_{i\in\sigma^*} \tau_i$, $j^* \in \tau$.

Suppose $T = \{(x^1_j)^*\}$. Then $(B^1_j)^* \wedge \bigwedge_{i\in\sigma} A^0_{\tau_i} \neq 0$, hence $j^* \in \bigcap_{i\in\sigma} \tau_i$, and this implies that χ is not a conjunct of ψ_1.

Suppose $T = \{y^1_\tau\}$. Then $\bigwedge_{i\in\sigma} A_1 \wedge A \neq 0$. So, there is $\sigma' \in \eta(A)$ such that $\sigma \subseteq \sigma'$. $\bigwedge_{i\in\sigma'} A^0_{\tau_i} \neq 0$, and since $K \vDash \varphi_1[A_{\tau_0}, \ldots, A_{\tau_{n-1}}]$ it follows that $\bigcap_{i\in\sigma'} \tau_i \neq \emptyset$. Let $j \in \bigcap_{i\in\sigma'} \tau_i$, hence $j \in \tau$ and $j \in \bigcap_{i\in\sigma} \tau_i$. This implies that χ is not a conjunct of ψ_1.

The case $T = \{(y^1_\tau)^*\}$ can be reduced to the previous case by replacing χ by χ^{*+}.

Suppose $T = \{(x^1_j)^*, y^1_\tau\}$. Then $\bigwedge_{i\in\sigma} A^0_{\tau_i} \wedge (B^1_j)^* \wedge A^1_\tau \neq 0$. This implies that $j^* \in \tau \cap \bigcap_{i\in\sigma} \tau_i$, hence χ is not a conjunct of ψ_1.

The last case that we check is $T = \{y^1_\tau, (y^1_\tau)^*\}$. Hence $\bigwedge_{i\in\sigma} A_i \wedge A \wedge A^* \neq 0$, so there is $D \in K$ such that $D \leqslant \bigwedge_{i\in\sigma} A_i \wedge A$ and $D^* \leqslant A$. Let $\sigma' \subseteq n$ and D' be such that $K \ni D' \subseteq D$, $D' \leqslant \bigwedge_{i\in\sigma'} A_i$ and for every $i \notin \sigma'$, $D' \wedge A_i = 0$. Hence $\sigma \subseteq \sigma' \in \eta(A)$, and since $(D')^* \leqslant A$, $(\sigma')^* \in \eta(A)$. Clearly, since $K \vDash \varphi_1[A^0_{\tau_0}, \ldots, A^0_{\tau_{n-1}}]$ it follows that there is $j \in \bigcap_{i\in\sigma'} \tau_i$; and since $\sigma \subseteq \sigma'$, $j \in \bigcap_{i\in\sigma} \tau_i$. Moreover $j^* \in \bigcap_{i\in(\sigma')^*} \tau_i$. Hence $j, j^* \in \tau$. These facts imply that χ is not a conjunct of ψ_1.

We leave the (easy) remaining cases to the reader. $\square$

Proof of Lemma 10.9. Let M be a model with universe $\aleph_1$ encoding all the information mentioned in the lemma. Suppose w.l.o.g. $t = 0$. For every C_M-slice E we decide how to divide the elements of E among the various B^0_i's. This is done independently of how the elements of other C_M-slices are divided. Let $\{a_m \mid m \in \omega\} = E$. Let F be the set of all real monotonic functions definable from ordinals $\alpha < \min(E)$. Note that F is countable and is closed under composition. We decide by induction on m to which B^0_i, a_m will belong. Hence at stage m we have sets $\{B^0_i(m) \mid i < n_0\}$. We denote $B_1(i)$ by $B^1_i(m)$. We assume by induction:

$$(*)\ \text{If}\quad \chi \equiv \bigwedge_{j\in\sigma_0} (x^0_j)^{\varepsilon(0,j)} \wedge \bigwedge_{j\in\sigma_1} (x^1_j)^{\varepsilon(1,j)} \wedge \bigwedge_{\tau\in\eta_0} (y^0_\tau)^{\varepsilon(0,\tau)} \wedge \bigwedge_{\tau\in\eta_1} (y^1_\tau)^{\varepsilon(1,\tau)} = 0$$

is a conjunct of ψ_1, then there is no $a \in E$, $\{f^l_j \mid l \in \{0,1\}, j \in \sigma_l\} \subseteq F$ and $\{f^l_j \mid l \in \{0,1\}, \tau \in \sigma_l\} \subseteq F$ such that: f^l_j and f^l_τ are OP or OR according to whether $\varepsilon(l,j)$ and $\varepsilon(l,\tau)$ are 0 or 1, $f^l_j(a) \in B^l_j(m)$ and $f^l_\tau(a) \in A^l_\tau$.

Let $B_i^0(0)=B_0(i)$. The induction hypotheses holds since

$$K\vDash\psi_1[B_0(i)\mid i<n_0;B_1(i)\mid i<n_1;A_\tau^0\mid\tau\in\Gamma_0;A_\tau^1\mid\tau\in\Gamma_1],$$

and since M encodes the above sets.

Suppose $B_0^0(m),\ldots,B_{n_0-1}^0(m)$ have been defined. Let

$$\tau_1=\bigcap\{\tau\mid a_m\in A_\tau^0\},$$
$$\tau_2^l=\bigcap\{\tau\mid(\exists f\in F)\,(f\text{ is OP and }f(a_m)\in A_\tau^l)\},\quad l=0,1,$$
$$\tau_3^l=\bigcap\{\tau\mid(\exists f\in F)\,(f\text{ is OR and }f(a_m)\in A_\tau^l)\},\quad l=0,1,$$
$$\tau_4^l=\{i\mid(\exists f\in F)\,(f\text{ is OP and }f(a_m)\in B_i^l(m))\},\quad l=0,1,$$
$$\tau_5^l=\{i\mid(\exists f\in F)\,(f\text{ is OR and }f(a_m)\in B_i^l(m))\},\quad l=0,1.$$

By the induction hypothesis each of the sets $\tau_4^0\cup\tau_4^1$, $\tau_5^0\cup\tau_5^1$ contains at most one element and if $i_j\in\tau_j^0\cup\tau_j^1$, $j=4,5$, and $i_4=(i_5)^*$. By the induction hypothesis

$$\tau\overset{\text{def}}{=}\tau_1\cap\bigcap_{l=0}^{1}\tau_2^l\cup\bigcap_{l=0}^{1}(\tau_3^l)^*\neq 0,$$

and if $i\in\tau_4^0\cup\tau_4^1$, then $i\in\tau$, and if $i\in\tau_5^0\cup\tau_5^1$, then $i^*\in\tau$. Let $i_0\in\tau_4^0\cup\tau_4^1$ if $\tau_4^0\cup\tau_4^1\neq 0$, $i_0\in(\tau_5^0\cup\tau_5^1)^*$ if $\tau_5^0\cup\tau_5^1\neq\emptyset$, and otherwise let i_0 be any member of τ. Let $B_i(m+1)=B_i^0(m)$ for every $i\neq i_0$ and $B_{i_0}^0(m+1)=B_{i_0}^0(m)\cup\{a_m\}$. It is easy to check that the induction hypothesis holds, and that the construction yields B_i^0's as required. □

Proof of 10.10. The construction of P resembles the forcing set constructed to prove Theorem 9.2. The proof that $\Vdash_P(K\vDash[B_0^0,\ldots,B_{n_0-1}^0,B_0^1,\ldots,B_{n_1-1}^1])$ resembles the proof of 9.6. We thus leave the details of the proof to the reader. □

This concludes the proof of 10.1.

On the possible infinite K^H's

We did not pursue a characterization of all possible K^H's, and not even all countably generated K^H's. However the construction of 10.1 can be applied to yield some new infinite K^H's. Also, some additional information about the structure of K and K^H can be derived from $\mathrm{MA}_{\aleph_1}$.

In the remainder of this section we first present some additional facts about the case of an infinite K^H, we then discuss some open problems, and finally we prove the theorems stated before,

Definition 10.11. (a) $A\in K$ is quasi-homogeneous (QH), if there is a family $\{A_i\mid i\in\omega\}\subseteq K^H$ such that for every $i\in\omega$, $A_i\subseteq A$ and $|A-\bigcup_{i\in\omega}A_i|\leqslant\aleph_0$.

(b) Let L be a σ-complete upper-semi-lattice, $a\in L$ is countably indecomposable (CID), if $a\neq 0$, and whenever $a\leqslant\bigvee_{i\in\omega}a_i$ there is $i\in\omega$ such that $a\leqslant a_i$. Let L^C denote the set of CID elements of L.

(c) Let $\langle M, \leqslant, 0\rangle$ be a poset with a smallest element 0. $A \subseteq M$ is dense in M, if for every $b \in M - \{0\}$ there is $a \in A - \{0\}$ such that $a \leqslant b$.

(d) A poset M is scattered if $\{\mathbb{Q}, <\rangle$ is not embeddable in M.

Theorem 10.12 ($\mathrm{MA}_{\aleph_1}$). (a) *Let $A \prec B$ mean that $A \preccurlyeq B$ and $A \not\equiv B$. If $\mathscr{A}$ generates K^H, $\mathscr{A}$ is countable and $\langle \mathscr{A}, \prec\rangle$ is well-founded, then K^{HC} generates K^H.*

(b) *If K^{HC} is dense in K^{HZ}, then K^{HC} is dense in $K \cup \{\emptyset\}$.*

(c) *Suppose K^{HC} is countable and scattered, and K^{HC} generates K^{HZ}, then every member of K is QH.*

Lemma 10.13. *Let $\langle M, \leqslant\rangle$ be a countable poset. Then up to isomorphism there is a unique complete lattice L with the following properties: $M = L^C$ and M generates L. In this unique lattice L the distributive law $b \wedge \bigvee_{i\in\omega} a_i = \bigvee_{i\in\omega}(b \wedge a_i)$ holds.*

Theorem 10.14. (a) *Let $\langle M, \leqslant, *\rangle$ be a countable poset with an involution with the following property: (*) "For every $A \subseteq M$: if for every $\tau \in P_{\aleph_0}(A)$ there is $b \in M$ such that for every $a \in \tau$ $b \leqslant a$, then there is $b \in M$ such that for every $a \in A$ $b \leqslant a$". Then*

$$\mathrm{CON}(\mathrm{MA}_{\aleph_1} + (\langle K^{HC}, \preccurlyeq, *\rangle \cong \langle M, \leqslant, *\rangle) + (K^{HC} \text{ generates } K^H) + (\text{every member of } K \text{ is QH})).$$

(b) *Let $V \vDash \mathrm{CH}$ and $\lambda \leqslant \kappa$ be cardinals in V. Then there is an extension W of V which has the same cardinals as V such that $W \vDash \mathrm{MA}_{\aleph_1} +$ "There is a family $\{A_i \mid i < \lambda\} \subseteq K$ such that for every $i \neq j$, $A_i \perp\!\!\!\perp A_j$ and for every $A \in K$, there is $i < \lambda$ such that $A_i \preccurlyeq A$" $+ 2^{\aleph_0} \geqslant \kappa$.*

(c) *It is consistent that $\mathrm{MA}_{\aleph_1}$ holds and $\langle K^{HZ}, \succ\rangle \cong \langle P_{\aleph_1}(\aleph_1) \cup \{\aleph_1\}, \subseteq\rangle$. It is consistent that $\mathrm{MA}_{\aleph_1}$ holds and $\langle K^{HZ}, \prec\rangle \cong \langle \aleph_1 + 1, <\rangle$.*

Let us now explain what seem to be the main open questions.

Question 10.15. It is easy to construct a universe satisfying

$$\mathrm{MA} + (\forall A \in K)\,(\exists B \in K)\,(B \subseteq A \text{ and } B \text{ is 2-entangled}).$$

In such a universe every $A \in K^H$ contains $B \in K^H$ such that every member of K^H contained in B is decomposable.

Construct a universe W satisfying $\mathrm{MA}_{\aleph_1}$ in which K^H is countably generated but K^{HC} does not generate K^H. Moreover, can W be constructed so that $K^{HC} = \emptyset$?

Question 10.16. Does the first or the second part of 2.14(c) remain true when $\aleph_1$ is replaced by some $\lambda > \aleph_1$?

Question 10.17. Is $\mathrm{MA}_{\aleph_1} + (K^{HC}$ is countable$) + (K^{HC}$ generates $K^H) + (\exists A \in K)\,(A$ is not QH$)$ consistent?

Question 10.18. Is the consistency result of Theorem 10.14(a) true when $\langle M, \leqslant\rangle$ is any countable poset?

Proof of Theorem 10.12. (a) The proof is easy.

(b) The following claim follows easily from 6.1(b).

Claim 1. *If $A \in K$, $B \in K^H$ and for every interval I of A, $B \leqslant I^m$, then $B \leqslant A$.*

Let us next prove the following claim.

Claim 2. *If $A \in K$, $B \in K^{HC}$ and $B \leqslant A^m$, then $B \leqslant A$.*

Proof. Let $A_1 = \bigcup\{I \mid I$ is an interval of A and $B \not\leqslant I^m\}$. By the countable indecomposability of B, $B \not\leqslant A_1^m$. Let $A_2 = A - A_1$, hence $A_2 \neq \emptyset$. Moreover, since $B \leqslant A^m$, for every interval J *of* A_2 $B \leqslant J^m$. By claim 1, $B \leqslant A_2 \subseteq A$. □

(b) follows easily from claim 2.

(c) **Claim 3.** *Suppose K^{HC} is countable and it generates K^H. Let $A \in K$, $B \in K^{HC}$ and $B \leqslant A^m$. Then there is $A_1 \subseteq A$ such that $A - A_1 \cong B$ and for every interval I of A_1 if $B \leqslant I$, then there is $B < C \in K^{HC}$ such that $C \leqslant I$.*

Proof. Let $A_2 = \bigcup\{I \mid I$ is an interval of A and $B \not\leqslant I\}$, $A_3 = A - A_2$ and $A_4 = \bigcup\{I \mid I$ is an interval of A_3 and for no $C \in K^{HC}$, $B < C \leqslant I\}$. Clearly there is $B' \cong B$ such that B' is a dense subset of A_3. If $|A_4| \leqslant \aleph_0$, let $A_1 = A - B'$; it is easy to see that A_1 is as required. Otherwise, it is easy to see that for some $\{B_i \mid i < \alpha \leqslant \omega\} \subseteq K^{HC}$, $A_4^m \cong \dot{\bigvee}_{i<\alpha} B_i$, $B_0 \cong B$ and for every $0 < i < \alpha$, $B \not< B_i$. Hence there is a countable family of OP functions $\mathscr{G}$ such that for every $g \in \mathscr{G}$ there is $i(g) < \alpha$ such that $g \subseteq B_{i(g)} \times A_4$, and $\bigcup\{\mathrm{Rng}(g) \mid g \in \mathscr{G}\} = A_4$. Let $B'' = B' \cup \bigcup\{\mathrm{Rng}(g) \mid g \in \mathscr{G}$ and $i(g) = 0\}$ and $A_1 = A - B''$. Since B' is dense in B'', $B'' \cong B$. Let I be an interval of A_1 and suppose that $B \leqslant I$. Clearly $B \leqslant I - A_2$, that is, $B \leqslant A_3 \cap I$. Suppose by contradiction there is no $C \in K^{HC}$ such that $B < C \leqslant I$. Let I' be the convex hull of $A_3 \cap I$ in A_3. Since $I' - I \leqslant B$ and $I - A_3 \not\leqslant B$ there is no $C \in K^{HC}$ such that $B < C \leqslant I'$. Hence $I' \subseteq A_4$. We will reach a contradiction if we show that $B \not\leqslant I' - B''$. Since $I' - B'' \subseteq \bigcup\{\mathrm{Rng}(g) \mid g \in \mathscr{G}$ and $(\mathrm{Dom}(g))^m \not\leqslant B\}$ it follows that $B \not\leqslant I' - B''$. Hence the claim is proved. □

Claim 4. *Suppose that $A \in K$, $B \in K^{HC}$, $B \leqslant A$ and for no $C \in K^{HC}$ $B < C \leqslant A$, then there is $B' \cong B$ such that $B \not\leqslant A - B'$.*

Proof. This is a special case of claim 3. □

Definition. Let $\langle M, \leqslant\rangle$ be a poset and let $\{M_i \mid i \in \omega\}$ be a family of subsets of M.

M is an *ω-sum* of $\{M_i \mid i \in \omega\}$ if $M = \bigcup_{i\in\omega} M_i$, and for every $i < j \in \omega$, $a \in M_i$ and $b \in M_j$, $b \nleq a$. We define ω^*-sums analogously.

Let $\mathcal{M}_0$ be the class of all posets with exactly zero or one element. For a limit ordinal δ let $\mathcal{M}_\delta = \bigcup_{\alpha<\delta} \mathcal{M}_\alpha$. Let $\mathcal{M}_{\alpha+1}$ be the class of all posets that can be represented as ω-sums or ω^*-sums of members of $\mathcal{M}_\alpha$. Let $\mathcal{M} = \mathcal{M}_{\aleph_1}$.

Claim 5. *Let M be a countable poset. Then M is scattered iff $M \in \mathcal{M}$.*

Proof. It is easy to show by induction on α that every member of M_α is scattered.

Let $\langle M, \leqslant\rangle$ be a scattered poset. By a theorem of Bonnet and Pouzet [5], there is a scattered linear ordering $\leqslant'$ on M extending $\leqslant$. Suppose M is countable, hence by a theoerem of Hausdorff there is $\alpha < \aleph_1$ such that $\langle M, \leqslant'\rangle \in \mathcal{M}_\alpha$. It is easy to see that $\langle M, \leqslant\rangle$ also belongs to $\mathcal{M}_\alpha$. $\square$

For $A \in K$ let $A^{HC} = \{B \in K^{HC} \mid B \subseteq A\}$. We assume that K^{HC} generates K^H. (c) follows from the following chain which is proved by induction on $\alpha < \aleph_1$.

Claim 6. *Let $A \in K$ and A^{HC} is a sum of K_1 and K_2, that is, $A^{HC} \subseteq K_1 \cup K_2$ and for every $B_1 \in K_1$ and $B_2 \in K_2$ $B_2 \nleq B_1$. Then, if $K_2 \in \mathcal{M}_\alpha$, then there is a QH set $A_2 \subseteq A$ such that $(A - A_2)^{HC} \subseteq K_1$.*

Proof. The case $\alpha = 0$ is just a reformulation of claim 4. There is nothing to prove for a limit ordinal α. Suppose the claim is true for α and we prove it for $\alpha + 1$. Let A^{HC} be the sum of K_1 and K_2, and $K_2 \in \mathcal{M}_{\alpha+1}$. Let us first deal with the case that K_2 is the ω^*-sum of $\{M_i \mid i \in \omega\}$ where each $M_i \in \mathcal{M}_\alpha$. Using the induction hypothesis we can define inductively $\{C_i \mid i \in \omega\}$ such that for every i, C_i is QH,

$$C_i \subseteq A - \bigcup_{j<i} C_j \quad \text{and} \quad \left(A - \bigcup_{j\leqslant i} C_j\right)^{HC} \subseteq K_1 \cup \bigcup_{j>i} M_j.$$

Let $A_2 = \bigcup_{i\in\omega} C_i$; it is easy to sse that A_2 is as required.

Let us assume that K_2 is an ω-sum of $\{M_i \mid i \in \omega\}$. $A^m = \bigcup\{rA + q \mid r, q \in \mathbb{Q}\}$. Let $f : \bigcup\{B \mid B \in K_1 \cup K_2\} \to A^m$ be an isomorphism. (We assume that each $B \in K_1 \cup K_2$ is dense in $\mathbb{R}$. For every $r, q \in \mathbb{Q}$ let $g_{r,g}(x) = (1/r)(x - q)$. For every $B \in K_1 \cup K_2$ let $f_{B,r,q} = g_{r,q} \circ (f \restriction B)$. Clearly, for every $B \in K_1 \cup K_2$, $f_{B,r,q} \subseteq B \times A$, and $\bigcup\{\mathrm{Rng}(f_{B,r,q}) \mid B \in K_1 \cup K_2$ and $r, q \in \mathbb{Q}\} = A$. For every $i \in \omega$, let $C_1 = \bigcup\{\mathrm{Rng}(f_{B,r,q}) \mid B \in K_1 \bigcup_{j\leqslant i} M_j, r, q \in \mathbb{Q}\}$. Hence $\bigcup_{i\in\omega} C_i = A$, and for every $i \in \omega$, $(C_i)^{HC} \subseteq K_1 \cup \bigcup_{j\leqslant i} M$. Since $\bigcup_{j\leqslant i} M_j$ can be regarded as an ω^*-sum (where some of the summands are empty), by the previous case there is a QH set $D_i \subseteq C_i$ such that $(C_i - D_i)^{HC} \subseteq K_1$.

Let $A_2 = \bigcup_{i\in\omega} D_i$. It is easy to see that A_2 is as required. $\square$

Proof of Lemma 10.13. *Existence.* Let $\sim$ be the following equivalence relation on $p(M)$: $M_1 \sim M_2$ if for every $m_1 \in M_1$ there is $m_2 \in M_2$ such that $m_1 \leqslant m_2$ and for every $m_2 \in M_2$ there is $m_1 \in M_1$ such that $m_2 \leq m_1$. Let $L = P(M)/\sim$. $M_1/\sim \leq M/\sim$ if for every $m_1 \in M_1$ there is $m_2 \in M_2$ such that $m_1 \leq m_2$. Clearly the definition of $\leq$ does not depend on the choice of representatives. It is easy to check that $\langle L, \leq \rangle$ is as required.

Uniqueness. For a lattice L_1 such that $L_1^C = M$ and L_1^C generates L_1 let $\varphi: L_1 \to L$ be defined as follows: $\psi(a) = \{m \in M \mid m \leq a\}/\sim$. It is easy to check that φ is an isomorphism between L_1 and L. □

Proof of Theorem 10.14(a). The proof of 10.14(a) resembles the proof of 10.1. However, some modifications have to be made. We skip those parts of the proof which are straight-forward generalizations of claims proved in 10.1. In order to simplify the technical details we deal with the special case in which the involution $*$ is the identity function. However, the proof is easily extended to the general case.

Lemma 10.19 (CH). *Let $\langle M, \leq \rangle$ be a countable poset. Then there is a family $\{A(m) \mid m \in \mathcal{M}\} \subseteq K^H$ such that* (1) *if $m \leq n$, then $A(m) \subseteq A(n)$;* (2) *$A(m) = A(m)^*$; and* (3) *let $B_m = A(m) - \bigcup_{n \lessdot m} A(n)$; then, if $m \nleq n$, then $B_m \perp\!\!\!\perp A(n)$.*

Proof. As in 10.5. □

For the rest of the proof $\langle M, \leq \rangle$ denotes a fixed countable poset with the property: $(*)$ "If $A \subseteq M$ and every finite subset of A has a lower bound, then A has a lower bound". We also fix some family $\{A(m) \mid m \in M\}$ as constructed in 10.19; $B_m = A(m) - \bigcup_{n \lessdot m} A(n)$.

Proposition 10.20. *If M is countable and has the property $(*)$, then M satisfies $(*)$ in every generic extension.*

Proof. Easy. □

We define F_m φ_0 and φ_1 as in 10.1. The induction hypotheses of the iteration are as in 10.1. So as in 10.7 at stage ν of the iteration we have sets $\{A(m) \mid m \in M\}$, $\{B(m) \mid m \in M\}$ and filters $\{F(m) \mid m \in M\}$, and we assume (1)–(6) of 10.7 hold.

Lemma 10.6 remains unchanged.

Definition 10.21. Let $\mathcal{A} \subseteq K^H$ and $D \in K$, D is $\mathcal{A}$-QH if there is $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}| \leq \aleph_0$, and for every $B \in \mathcal{B}$ there is $C(B) \cong B$ such that $C(B) \subseteq D$ and $|D - \bigcup \{C(B) \mid B \in \mathcal{B}\}| \leq \aleph_0$.

Proposition 10.22 ($MA_{\aleph_1}$). *If $\mathscr{A} \subseteq K^H$ and every $D \in K$ is $\mathscr{A}$-QH, then $\mathscr{A}$ generates K^H.*

Proof. Trivial. □

For every $m \in M$, let $\tau_m = \{n \mid n \leqslant m\}$ and $A(\tau_m) = A(m)$. The following lemma is the counterpart of 10.7.

Lemma 10.23 (CH). *Let $A(m)$, $B(m)$ and $F(m)$, $m \in M$, satisfy (1)–(6), and let $A \in K$. Then there are $B'(m) \in F(m)$ and a c.c.c forcing set P of power $\aleph_1$ such that*

$$\Vdash_P (K \vDash \varphi_1[B'(m) \mid m \in M; A(\tau_m) \mid m \in M]) \wedge (A \text{ is } \{A(m) \mid m \in M\}\text{-}QH).$$

It is obvious that 10.14(a) follows from 10.19, the analogue of 10.6, 10.20 and 10.21. We now formulate the analogues of 10.8–10.10. Let ψ_0 and ψ_1 be as in 10.1.

Lemma 10.24 (CH). *Let $A(m)$, $B(m)$, $F(m)$, $m \in M$, satisfy (1)–(6), and let $A \in K$. Then there is $\tau \subseteq M$, $\{B_1(m) \mid m \in \tau\}$ and $\{B_0(m) \mid m \in M\}$ such that:*

(1) *If $n \leqslant m \in \tau$, then $n \in \tau$.*

(2) *For every $m \in \tau$, $K \ni B_1(m) \subseteq A$.*

(3) *For every $m \in M$, $B_0(m) \in F(m)$.*

(4) *Let $\tau_m = \{n \mid n \leqslant m\}$, $A_0(\tau_m) \stackrel{\text{def}}{=} A(m)$ and $A_1(\tau) \stackrel{\text{def}}{=} A$; then $K \vDash \psi_1[B_0(m) \mid m \in M;\ B_1(m) \mid m \in \tau;\ A_0(\tau_m) \mid m \in M;\ A_1(\tau)]$.*

(5) *For every $n \in \tau$ and $B \in K$, if $B_1(n) \subseteq B \subseteq A$ and $K \vDash \psi_1[B_0(m) \mid m \in M; B; A_0(\tau_m) \mid m \in M; A_1(\tau)]$, then $B_1(n)$ is dense in B.*

(6) *If $n \leqslant m \in \tau$, then $B_1(n)$ is dense in $B_1(n) \cup B_1(m)$.*

Lemma 10.25 (CH). *For $l = 0, 1$ let $M_l \subseteq M$, $\Gamma_l \subseteq P(M_l)$, $\{B_l(m) \mid m \in M_l\} \subseteq K$ and $\{A_l(\tau) \mid \tau \in \Gamma_l\} \subseteq K$. Suppose the following conditions hold:*

(1) *If $m, n \in M_l$ are distinct, then $B_l(m) \cap B_l(n) = \emptyset$, and if $m \in \tau \in \Gamma_l$, then $B_l(i) \subseteq A_\tau^l$.*

(2) *If $\tau_1, \tau_2 \in \Gamma_l$ and $\tau_1 \subseteq \tau_2$, then $A_l(\tau_1) \subseteq A_l(\tau_2)$.*

(3) *If $\Gamma \subseteq \Gamma_l$ and $\bigcap \Gamma = \emptyset$, then for some finite $\Gamma' \subseteq \Gamma$, $\bigcap \Gamma' = \emptyset$.*

(4) *$K \vDash \psi_1[B_l(m) \mid l \in \{0, 1\},\ m \in M_l;\ A_l(\tau)\ l \in \{0, 1\}, \tau \in \Gamma_l]$.*

Let $t \in \{0, 1\}$. Then there are pairwise disjoint $\{B^t(m) \mid m \in M_t\}$ such that

(1) *$K \ni B^t(m) \supseteq B_t(m)$.*

(2) *For every $\tau \in \Gamma_t$, $A_t(\tau) \supseteq \bigcup_{m \in \tau} B^t(m)$.*

(3) *$K \vDash \psi_1[B^t(m) \mid m \in M_t;\ B_{1-t}(m) \mid m \in M_{1-t};\ A_{1-t}(\tau) \mid \tau \in \Gamma_{1-t}]$.*

Lemma 10.26 (CH). *Let $M_0, M_1 \subseteq M$, $L \subseteq M_0 \subseteq M_1$, $\{B^l(m) \mid l < 2, m \in M_l\} \subseteq K$ and $\{D^l(m) \mid l < 2,\ m \in L\} \subseteq K$ be such that:*

(1) *$m \neq n$ implies $B^l(m) \cap B^l(n) = \emptyset$.*

(2) *$D^l(m) \subseteq B^l(m)$.*

(3) *For every* $m \in L$, $D^l(m)$ *is dense in* $\bigcup\{D^l(n) \mid n \in L\}$.
(4) $K \models \psi_0[B^l(m) \mid l<2, m \in M_l]$.

Then there is a c.c.c. forcing set P of power $\aleph_1$ *such that*

$$\Vdash_P \Big(\bigcup_{m\in L} D^0(m) \cong \bigcup_{m\in L} D^1(m)\Big) \wedge \Big(K \models \psi_0[B^l(m) \mid l<2, m \in M_l]\Big).$$

The argument why 10.23 follows from 10.24–10.26, resembles the analogous argument in 10.1. Also 10.26 is the same as 10.10. We also omit the proof of 10.25, since it involves no new difficulties.

Proof of 10.24. Let $A(m)$, $B(m)$, $F(m)$ and A be as in 10.24. Let $S(m) = \{B \subseteq B(m) \mid (\forall B' \in F(m))(B \cap B' \neq \emptyset)\}$. Let $K \ni C \subseteq A$ and $m \in M$. We say C is appropriate for m if: either m is a minimal element of M, for every $n \in M$ such that $n \not\geq m$, $C \perp\!\!\!\perp A(n)$, and for every $n \neq m$ and $B \in S(n)$, $B \not\leqslant C$; or if for some $B \in S(m)$, $C \cong B$. Note that if C is appropriate for m, then there is $B_0(m) \in F(m)$ such that $K \models \psi_1[s]$ where s is the assignment in which for every $n \neq m$, $s(x^0_n) = B(n)$, $s(x^0_m) = B_0(m)$, $s(x^1_m) = C$ and for every $m \in M$, $s(y^0_{\tau_m}) = A(m)$.

Let $\tau = \{m \in M \mid (\exists C \subseteq A)(\exists m' \geqslant m)$ (C is appropriate for m')$\}$. W.l.o.g. A is dense in $\mathbb{R}$. For every interval I of A with rational endpoints, let $\tau_I = \{m \in M \mid (\exists C \subseteq I)$ (C is appropriate for m)$\}$ and for every $m \in \tau_I$, let $C(I, m) \subseteq I$ be appropriate for m. For every $n \in \tau$, let $C(n) = \bigcup\{C(I, m) \mid m \in \tau_I$ and $n \leqslant m\}$. It is already standard to construct $\{B_0(m) \mid m \in M\}$, $\{B_1(m) \mid m \in \tau\} \subseteq K$ such that: (1) $B_0(m) \in F(m)$; (2) $B_1(m)$ is a dense subset of $C(m)$; and (3) for every $m \neq n$, $B_1(m) \perp\!\!\!\perp (B_1(n) \cup B_0(n))$. (Here we assume that $B_1(n) = \emptyset$ if $n \notin \tau$.)

We check that τ, $\{B_0(m) \mid m \in M\}$ and $\{B_1(m) \mid m \in M\}$ satisfy the requirements of the lemma. Requirements (1)–(3) are automatically satisfied, (5) and (6) are easily checked. We deal with (4). As in the proof of 10.8, we have to prove the following claim. Let $\chi \equiv \bigwedge_{m\in\sigma} y^0_{\tau_m} \wedge \bigwedge\{t \in T\}$ where

$$T \subseteq \{x^0_m \mid m \in M\} \cup \{x^1_m \mid m \in \rho\} \cup \{(x^1_m)^* \mid m \in \tau\} \cup \{y^1_\tau, (y^1_\tau)^*\}$$

and T intersects the union of the first three sets in at most one element; and let s be the assignment such that $s(x^l_m) = B^l(m)$, $s(y^0_{\tau_m}) = A(m)$ and $s(y^1_\tau) = A$. *Then* if $K \models \neg\chi[s]$, then χ is not a conjunct of ψ_1.

In fact, the above claim has to be proved just for χ's in which σ is finite. This follows easily from the fact: (*) "For every $L \subseteq M$, if every finite $L_0 \subseteq L$ has a lower bound, then L has a lower bound".

Of the many cases in 10.8 we check only that case which calls for a new argument. Let $\chi \equiv (\bigwedge_{m\in\sigma} y^0_{\tau_m}) \wedge y^1_\tau$ and suppose that $K \models \neg\chi[s]$. We have to prove that χ is not a conjunct of ψ_1, that is, we have to prove that there is $n \in \tau$ such that for every $m \in \sigma$, $n \leqslant m$. $K \models \neg\chi[s]$ means that $A \wedge \bigwedge_{m\in\sigma} A(m) \neq 0$, hence let $C \in K$ be such that $C \leqslant A$, and $C \leqslant A(m)$ for every $m \in \sigma$. If for some $n \in M$ and some $B \in S(m)$, $B \leqslant C$, then $n \in \tau$ and $n \leqslant m$ for every $m \in \sigma$, hence we are through. Suppose the above does not happen. We prove the following claim.

Claim. *There is $K \ni D \leqslant C$ such that every finite subset of $L = \{m \mid \neg(A(m) \perp\!\!\!\perp D)\}$ has a lower bound.*

Proof. Suppose the above claim is not true. We define a tree $\langle T, \leqslant,\rangle \subseteq {}^{\omega>}\omega$, and for every $\nu \in T$ we define $m(\nu) \in M$ and $C_\nu \in K$ such that: (1) every member of T has at least two successors; (2) for every $\nu \in T$, $C_\nu \subseteq C$; (3) if $\nu \leqslant \xi$, then $m(\xi) \leqslant m(\nu)$; and (4) if $\nu \in T$ and $\{\xi_0, \ldots, \xi_{r-1}\}$ is the set of successors of ν in T, then $\{m(\xi_0), \ldots, m(\xi_{r-1})\}$ does not have a lower bound in M. The construction is done easily by induction.

We show that there is a branch $\{\nu_i \mid i \in \omega\}$ of T such that $\{m(\nu_i) \mid i \in \omega\}$ does not have a lower bound. Let $\{m_i \mid i \in \omega\}$ be an enumeration of M. Let $\nu_0 = \Lambda$. Suppose ν_i has been defined. By (4) there is a successor ξ of ν_i such that $m_i \not\leqslant m(\xi)$. Let $\nu_{i+1} = \xi$. Hence $\{m(\nu_i) \mid i \in \omega\}$ does not have a lower bound, however every finite subset of $\{m(\nu_i) \mid i \in \omega\}$ has a lower bound. This contradicts property (∗) of M. This concludes the proof of the claim. □

Let D be as assured in the claim, and let L be as defined in the claim, and let n be a lower bound for L. We check that D is appropriate for n. By our assumption for every $m \in M$ and $B \in S(m)$, $B \not\leqslant D$, and it follows from the properties of L and D that if $\neg(A(m) \perp\!\!\!\perp D)$, then $n \not\leqslant m$. Hence $n \in \tau$. For every $m \in \sigma$, $D \not\leqslant A(m)$, hence $m \in L$ and so $n \leqslant m$. We have thus found $n \in M$ such that $n \in \tau$ and for every $m \in \sigma$, $n \leqslant m$. This concludes the proof of 10.24 and the proof of 10.14(a). □

We leave the proofs of 10.14(b) and (c) to the reader, since they do not involve any new difficulties.

11. MA + OCA implies $2^{\aleph_0} = \aleph_2$

In this short section we show that MA + OCA $\Rightarrow 2^{\aleph_0} = \aleph_2$. This fact follows from the following theorem.

Theorem 11.1 (ZFC). *There is a c.c.c. forcing set P of power $\aleph_2$ and a family $\{D_\nu \mid \nu < \aleph_2\}$ of dense subsets of P such that if $G \in V$ is a filter of P which intersects every D_ν, $\nu < \aleph_2$, then V contains a set $X \subseteq \mathbb{R}$ and an open coloring $\mathcal{U} = \{U_0, U_1\}$ of X such that $|X| = \aleph_1$ and X cannot be partitioned into countably many $\mathcal{U}$-homogeneous sets.*

Clearly, Theorem 11.1 implies that MA + OCA $\Rightarrow 2^{\aleph_0} = \aleph_2$, for if $V \models \text{Ma} + 2^{\aleph_0} > \aleph_2$, then V contains a filter $G \subseteq P$ which intersects the D_ν's, the thus V contains a coloring refuting OCA.

Moreover, Theorem 11.1 shows that it is consistent that $V \models \text{A1}$, but still if

$W \supseteq V$ and $W \vDash \mathrm{MA}+\mathrm{OCA}$, then $\aleph_1^{(V)}$ or $\aleph_2^{(V)}$ are collapsed. This is true for the universe V which is obtained in the following way. Let $V_0 \vDash \mathrm{CH}$, and let $V = V_0[P_{\mathrm{Cb}}(\aleph_3)][P]$ where P is the forcing set of 11.1.

The proof of 11.1 is divided into three claims.

Lemma 11.2. *There is a symmetric function $F \in V$, $F: \aleph_2 \times \aleph_2 \to \aleph_1$ such that for every universe $W \supseteq V$ and $A \in W$: if $\aleph_1^V = \aleph_1^W$, $\aleph_2^V = \aleph_2^W$, $A \subseteq \aleph_2$ and $|A| = \aleph_2$, then $F(A \times A)$ is unbounded in $\aleph_1$.*

Proof. For every $\aleph_1 \leqslant \nu < \aleph_2$, let $\{a(\nu, \alpha) \mid \alpha < \aleph_1\}$ be a 1-1 enumeration of ν, and for $\xi \leqslant \nu < \aleph_2$, let $F(\xi, \nu) = 0$ if $\xi = \nu$ or if $\nu < \aleph_1$, and let $F(\xi, \nu) = \alpha$ if $\xi = a(\nu, \alpha)$. It is easy to check that F is as required. □

We reserve the symbol F to denote a function as in 11.2. Recall that for a set A, $D(A) = A \times A - \{\langle a, a \rangle \mid a \in A\}$.

Lemma 11.3. *Let $A = \{a_\alpha \mid \alpha < \aleph_1\} \subseteq {}^\omega 2$, $\mathcal{U} = \{U_0, U_1\}$ be a partition of $D(A)$ into symmetric open sets, and let $\{H_\nu^l \mid l = 0, 1, \nu < \aleph_2\}$ be such that: for every $\nu < \aleph_2$ and $l \in \{0, 1\}$, $D(H_\nu^l) \subseteq U_l$, and for every $\nu, \xi < \aleph_2$ there is $\alpha(\nu, \xi) < \aleph_1$ such that $H_\nu^0 \cap H_\xi^1 = \{a_{\alpha(\nu,\xi)}\}$ and $\alpha(\nu, \xi) \geqslant F(\nu, \xi)$. Then A cannot be partitioned into countably many $\mathcal{U}$-homogeneous subsets.*

Proof. Suppose by contradiction $\{A_i \mid i \in \omega\}$ is a partition of A into $\mathcal{U}$-homogeneous sets, and let $\varepsilon(i)$ be such that $D(A_i) \subseteq U_{\varepsilon(i)}$.

For every $\nu < \aleph_2$ let

$$\beta(\nu) = \mathrm{Sup}(\{\alpha \mid (\exists l \in \{0, 1\}) (\exists i \in \omega) (\varepsilon(i) = l \wedge a_\alpha \in A_i \cap H_\nu^{1-l})\}).$$

$\beta(\nu)$ is a supremum of a countable set, hence $\beta(\nu) < \aleph_1$. Let $\Gamma \subseteq \aleph_2$ and $\beta_0 < \aleph_1$ be such that $|\Gamma| = \aleph_2$ and for every $\nu \in \Gamma$, $\beta(\nu) = \beta_0$. By the property of F there are $\nu, \xi \in \Gamma$ such that $F(\nu, \xi) > \beta_0$. Hence $\alpha_0 \stackrel{\mathrm{def}}{=} \alpha(\nu, \xi) > \beta_0$. Suppose $\alpha(\nu, \xi) \in A_i$. If $\varepsilon(i) = 0$, then the fact that $a_{\alpha_0} \in A_i \cap H_\xi^1$ implies that $\beta(\xi) \geqslant \alpha_0 > \beta_0$; and if $\varepsilon(i) = 1$, then the fact that $a_{\alpha_0} \in A_i \cap H_\nu^0$ implies that $\beta(\nu) \geqslant \alpha_0 > \beta_0$. In both cases we obtain that for some $\zeta \in \Gamma$, $\beta(\zeta) > \beta_0$ contradicting the choice of Γ. □

Lemma 11.4 (ZFC). *There is a c.c.c. forcing set P of power $\aleph_2$ and a family $\{D_\nu \mid \nu < \aleph_2\}$ of dense subsets of P such that if V contains a filter of P which intersects every D_ν, $\nu < \aleph_2$, then V contains a system $A = \{a_\alpha \in \alpha < \aleph_1\}$, $\mathcal{U} = \{U_0, U_1\}$ and $\{H_\nu^l \mid l = 0, 1, \nu < \aleph_2\}$* as in 11.3.

Proof. Let F be as assured by Lemma 11.2. We first define P. Each element p of P is a triple $\langle \mathcal{U}(p), g(p), f(p) \rangle$ where:

(1) $\mathcal{U}_p = \langle U(p, 0), U(p, 1) \rangle$ is a pair of disjoint symmetric clopen subsets of ${}^\omega 2$ such that $U(p, 0)$, $U(p, 1) \subseteq D({}^\omega 2)$. $\mathcal{U}_p$ is an approximation of the coloring $\mathcal{U}$ we wish to construct.

(2) $g(p)$ is a function such that $\mathrm{Dom}(g(p))=\sigma(p,0)\times\sigma(p,1)$ where the $\sigma(p,l)$'s are finite subsets of $\aleph_2$, $\mathrm{Rng}(g(p))\subseteq\aleph_1$, and for every $\langle\omega,\xi\rangle\in\mathrm{Dom}(g(p))$, $F(\nu,\xi)\leq g(p)(\nu,\xi)<F(\nu,\xi)+\omega$. We denote $g(p)(\nu,\xi)$ by $g(p,\nu,\xi)$. $g(p)$ is a finite approximation of the function $\alpha(\nu,\xi)$ of 11.3, that is, $g(p,\nu,\xi)=\alpha$ will mean that $\{a_\alpha\}=H^0_\nu\cap H^1_\xi$.

(3) $f(p)$ is a function such that $\sigma(p)\overset{\text{def}}{=}\mathrm{Dom}(f(p))$ is a finite subet of $\mathrm{Rng}(g(p))\times\omega$, and $\mathrm{Rng}(f(p))\subseteq\{0,1\}$. $f(p)$ is a finite information about the reals a_α where $\alpha\in\mathrm{Rng}(g(p))$, that is $f(p)(\alpha,n)=0$ will mean that for the real a_α of 11.3, $a_\alpha(n)=0$. We denote $f(p)(\alpha,n)=f(p,\alpha,n)$.

A triple $p=\langle\mathcal{U}(p),g(p),f(p)\rangle$ as above belongs to P if:

(1) For every distinct $\alpha,\beta\in\mathrm{Rng}(g(p))$, $f(p)$ already determines the $\mathcal{U}(p)$-color of $\langle a_\alpha,a_\beta\rangle$, that is, if for every $\gamma<\aleph_1$ we denote by $f(p,\gamma)$ the function such that for every $n\in\omega$, $f(p,\gamma)(n)=f(p,\gamma,n)$, then there is $l\in\{0,1\}$ such that

$$U(p,\alpha,\beta)\overset{\text{def}}{=}\{\langle a,b\rangle\in{}^\omega2\mid a\supseteq f(p,\alpha),\,b\supseteq f(p,\beta)\}\subseteq U(p,l).$$

(2) If $\alpha_1,\alpha_2\in\mathrm{Rng}(g(p))$, are distinct and for some ν,ξ_1,ξ_2, $\alpha_i=g(p,\nu,\xi_i)$, then the coloring of $\langle a_{\alpha_1},a_{\alpha_2}\rangle$ is determined to be 0, that is $U(p,\alpha_1,\alpha_2)\subseteq U(p,0)$, and if $\alpha_i=g(p,\xi_i,\nu)$, then the coloring of $\langle a_{\alpha_1},a_{\alpha_2}\rangle$ is determined to be 1, that is $U(p,\alpha_1,\alpha_2)\subseteq U(p,1)$.

$p\leq q$ if $U(p,l)\subseteq U(q,l)$, $l=0,1$, $g(p)\subseteq g(q)$ and $f(p)\subseteq f(q)$. This concludes the definition of P.

We leave it to the reader to check that by means of a family $\{D_\nu\mid\nu<\aleph_2\}$ of dense subsets of P one can assure the existence of a system $A,\mathcal{U},\{H^l_\nu\mid l\in\{0,1\},\nu<\aleph_2\}$ as required. We now turn to the proof that P is c.c.c.

Let $\{p^0_\alpha\mid\alpha^0<\aleph_1\}\subseteq P$. We uniformize $\{p_\alpha\mid\alpha<\aleph_1\}$ as much as possible. Hence we can assume that for some $\mathcal{U}=\langle U_1,U_2\rangle$ for every $\alpha<\aleph_1$, $\mathcal{U}(p_\alpha)=\mathcal{U}$, and that $\{\sigma_l(p_\alpha)\mid\alpha<\aleph_1\}$ are Δ-systems for $l=0,1$. Let $\sigma_l(p_\alpha)=\{\nu(\alpha,l,0),\ldots,\nu(\alpha,l,n_l-1)\}$, and $m_l\leq n_l$ be such that for every $i<m_l$ and $\beta,\gamma<\aleph_1$, $\nu(\beta,l,i)=\nu(\gamma,l,i)$. We assume that for every $i_0<n_0$ and $i_1<n_1$ either all the $g(p_\alpha,\nu(\alpha,0,i_0),\nu(\alpha,1,i_1))$'s are pairwise distinct, or they are all equal. Finally we assume that letting $\gamma(\alpha,i,j)$ denote $g(p_\alpha,\nu(\alpha,0,i),\nu(\alpha,1,j))$, for every $\alpha,\beta<\aleph_1$, $i_0<n_0$ and $i_1<n_1$: $f(p_\alpha,\gamma(\alpha,i_0,i_1))=f(p_\beta,\gamma(\alpha,i_0,i_1))$.

We prove that for every α and β, p_α and p_β are compatible. Let $q'=\langle\mathcal{U},g(p_\alpha)\cup g(p_\beta),f(p_\alpha)\cup f(p_\beta)\rangle$. q' is not a condition, but we prove that q' can be extended to a condition q. First we check that q' does not contain contradictions. Since $f(p_\alpha,\gamma(\alpha,i_0,i_1))=f(p_\beta,\gamma(\alpha,i_0,i_1))$ for every i_0 and i_1, and since U_1 and U_2 do not intersect the diagonal of ${}^\omega2\times{}^\omega2$, $\mathcal{U}$ determines the color of $\langle a_{\gamma(\alpha,i_0,i_1)},a_{\gamma(\beta,j_0,j_1)}\rangle$ iff $\langle i_0,i_1\rangle\neq j_0,j_1\rangle$, and if $\langle i_0,i_1\rangle\neq\langle j_0,j_1\rangle$, then the color of the above pair is equal to the color of $\langle a_{\gamma(\alpha,i_0,i_1)},a_{\gamma(\alpha,j_0,j_1)}\rangle$. This implies that q' makes no mistakes in determining colors.

Let $g'=g(p_\alpha)\cup g(p_\beta)$, and $\sigma_l=\sigma_l(p_\alpha)\cup\sigma_l(g_\beta)$, $l=0,1$. $\mathrm{Dom}(q')\neq\sigma_0\times\sigma_1$, hence we have to define $g\supseteq g'$ such that $\mathrm{Dom}(g)=\sigma_0\times\sigma_1$. Since for every

$\nu, \xi < \aleph_2$ there are $\aleph_0$ options of how to define $g(\nu, \xi)$ and since Rng(g') is finite for every $\langle \nu, \xi \rangle \in \sigma_0\sigma_1 - \mathrm{Dom}(g')$ we can find $g(\nu, \xi) \in [F(\nu, \xi), F(\nu, \xi)+\omega) - \mathrm{Rng}(g')$ so that $g - g'$ is 1-1. We leave it to the reader to check that $\mathcal{U}$ and $f(p_\alpha) \cup f(p_\beta)$ can be extended so that the color of every pair in $\{a_\gamma \mid \gamma \in \mathrm{Rng}(g)\}$ will be determined, and every such pair will have the right color. We thus constructed a condition extending p_α and p_β, hence p_α and p_β are compatible. □

Discussion. Let A denote the axiom $\mathrm{MA} + 2^{\aleph_0} > \aleph_2$. We found that A is not consistent with OCA, on the other hand it is consistent with BA, SOCA, NWD2, RHA and many other axioms whose consistency can be proved with the aid of the club method. There are two cases in which we do not know the answer to such a question.

Question 11.5. (a) Is A consistent with SOCA1?

(b) For which (finite) lattices with involution L is $\mathrm{A} + (K^H \cong L)$ consistent?

Question 11.5(a) is related to the following questions:

Question 11.6. (a) Is ¬CH consistent with the following axiom: "If B is an uncountable set of reals and $\{A_i \mid i \in I\}$ is a family of nwd subsets of B, which contains all finite subsets of B, and such that B is not contained in a union of countably many A_i's, then there is an uncountable $B' \subseteq B$ such that B' intersects each A_i in at most $\aleph_0$ points"?

(b) Is there a universe V such that $V \models 2^{\aleph_1} > \aleph_2$ and such that for every c.c.c. forcing set P of power $< 2^{\aleph_1}$, V^P has the following property: "If $\{A_i \mid i \in I\}$ is a family of $< 2^{\aleph_1}$ subsets of $\aleph_1$ such that $\aleph_1$ is not the union of any countably many A_i's, then there is an uncountable $A \subseteq \aleph_1$ which intersects each A_i in at most $\aleph_0$ points"?

Question 11.7. Is OCA consistent with $2^{\aleph_0} > \aleph_2$?

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