Invariant σ-ideals with analytic base on good Cantor measure spaces

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Consider $(2^\omega, \lambda)$, where λ is the standard product measure on 2^ω.

- $\mathcal{I} \subseteq P(2^\omega)$ is a σ-ideal if
  - $A \in \mathcal{I}$ implies that $P(A) \subseteq \mathcal{I}$,
  - $A_0, A_1, A_2, \ldots \in \mathcal{I}$ imples that $\bigcup_{n\in\omega} A_n \in \mathcal{I}$.
- $\mathcal{I}$ has Borel (analytic) base, if $(\forall A \in \mathcal{I})(\exists B \in \mathcal{I})(A \subseteq B \wedge B$ is Borel (analytic))
- $\mathcal{I}$ is invariant, if $\mathcal{I}$ is invariant under maesure-preserving homeomrphisms, i.e $(\forall h \in \mathcal{H}_\lambda(2^\omega))(\forall A \in \mathcal{I})(h[A] \in \mathcal{I})$.
- $\mathcal{I}$ is nontrivial, if $2^\omega \notin \mathcal{I}$ and $\mathcal{I}$ contains an uncountable set.

Problem.
Classify all nontrivial invariant σ-ideals $\mathcal{I}$ having Borel base.

Definition.

(X, μ) is called a Cantor measure space if the topological space X is homeomorphic to the Cantor cube $\{0,1\}^\omega$ and the measure $\mu : \mathcal{B}(X) \to [0, \infty)$ is *continuous* in the sense that $\mu(\{x\}) = 0$ for any point $x \in X$.

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Example.

There is a Cantor measure space (X, μ) that supports $2^\mathfrak{c}$ pairwise distinct invariant σ-ideals with Borel base.

E.Akin, *Good measures on Cantor space*, Trans. Amer. Math. Soc. **357**:7 (2005), 2681–2722.

Definition.

A Cantor measure space (X, μ) is called good if its measure μ is *good* in the sense of Akin, i.e.,

- *strictly positive* (which means that $\mu(U) > 0$ for any non-empty open set $U \subseteq X$),
- μ satisfies the *Subset Condition* which means that for any clopen sets $U, V \subseteq X$ with $\mu(U) < \mu(V)$ there is a clopen set $U' \subseteq V$ such that $\mu(U') = \mu(U)$.

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- All infinite compact metrizable zero-dimensional topological groups G endowed with the Haar measure are good Cantor measure spaces.
- By Akin's Theorem,

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a Cantor measure space (X, μ) is isomorphic to a (monothetic) compact topological group G endowed with the Haar measure if and only if (X, μ) is good and $1 \in \mu[\mathrm{Clop}(X)] \subseteq \mathbb{Q} \cap [0, 1]$.

On each measure space (X, μ) consider the following four invariant σ-ideals with Borel base:

- the σ-ideal $\mathcal{M}$ of meager subsets of X (it is generated by closed nowhere dense subsets of X);
- the σ-ideal $\mathcal{N} = \{A \subseteq X : \mu(A) = 0\}$ of null subsets of (X, μ) (it is generated by Borel subsets of zero μ-measure);
- the σ-ideal $\mathcal{M} \cap \mathcal{N}$ of meager null subsets of (X, μ);
- the σ-ideal $\mathcal{E}$ generated by closed null subsets of (X, μ).

Main theorem

Each non-trivial invariant σ-ideal $\mathcal{I}$ with analytic base on a good Cantor measure space (X, μ) is equal to one of the σ-ideals:

$$\mathcal{E},\ \mathcal{M} \cap \mathcal{N},\ \mathcal{M} \text{ or } \mathcal{N}.$$

Lemma 1. (Akin)

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Two good Cantor measure spaces (X, μ) and (Y, λ) are isomorphic if and only if $\mu[\mathrm{Clop}(X)] = \lambda[\mathrm{Clop}(Y)]$.

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Lemma 2.

Let (X, μ) be a good Cantor measure space, $U \subseteq X$ be a clopen set and $K \subseteq U$ be a compact subset. For every $\alpha \in \mu[\mathrm{Clop}(X)]$ with $\mu(K) < \alpha \leq \mu(U)$ there is a clopen subset $V \subseteq U$ such that $K \subseteq V$ and $\mu(V) = \alpha$.

Lemma 3. (Ryll-Nardzewski)

Any homeomorphism $f : A \to B$ between closed nowhere dense subsets $A, B \subseteq X$ of the Cantor cube $X = \{0,1\}^\omega$ extends to a homeomorphism $\bar{f} : X \to X$ of X.

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Lemma 4.

Any measure-preserving homeomorphism $f : A \to B$ between closed nowhere dense subsets $A, B \subseteq X$ of a good Cantor measure space (X, μ) extends to a measure-preserving homeomorphism $\bar{f} : X \to X$ of X.

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Proposition 1.

$\mathcal{E} \subseteq \mathcal{I}$.

Lemma 5.

If an analytic subset $A \subseteq X$ of a Cantor measure space (X, μ) is not contained in the σ-ideal $\mathcal{E}$, then A contains a G_δ-subset G of X such that $\mu(G) = 0$ and the measure $\mu \restriction \bar{G}$ is strictly positive.

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Lemma 6.

If $A \subseteq X$ is a closed subset of positive measure in a good Cantor measure space (X, μ), then for any $\varepsilon > 0$ there are homeomorphisms $h_1, \dots, h_n \in \mathcal{H}_\mu(X)$ such that the set $B = \bigcup_{i=1}^n h_i[A]$ has measure $\mu(B) > \mu(X) - \varepsilon$.

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Lemma 7.

Let (X, μ), (Y, λ) be Cantor measure spaces such that $\mu(X) < \lambda(Y)$ and the measure λ is strictly positive. Let $G_X \subseteq X$ and $G_Y \subseteq Y$ be two G_δ-sets of measure $\mu(G_X) = \lambda(G_Y) = 0$ such that G_Y is dense in Y. Then there is a measure-preserving embedding $f : X \to Y$ such that $f[G_X] \subseteq G_Y$.

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Proposition 2.

If $\mathcal{I} \not\subseteq \mathcal{E}$, then $\mathcal{M} \cap \mathcal{N} \subseteq \mathcal{I}$.

Lemma 8.

Let (X, μ) be a good Cantor measure space, A be a closed nowhere dense subset and $B \subseteq X$ be a Borel subset of measure $\mu(B) > \mu(A)$ in X. Then there is a measure-preserving homeomorphism $h : X \to X$ such that $h[A] \subseteq B$.

Lemma 8.
Let (X, μ) be a good Cantor measure space, A be a closed nowhere dense subset and $B \subseteq X$ be a Borel subset of measure $\mu(B) > \mu(A)$ in X. Then there is a measure-preserving homeomorphism $h : X \to X$ such that $h[A] \subseteq B$.

Proposition 3.
If $\mathcal{I} \not\subseteq \mathcal{N}$, then $\mathcal{M} \subseteq \mathcal{I}$.

Lemma 9.

Let (X, μ) be a good Cantor measure space, and d be a metric generating the topology of X. Let $B \subseteq X$ be a Borel subset of measure $\mu(B) = \mu(X)$. Let $A \subseteq C$ be two closed nowhere dense subsets in X such that $A \subseteq B$. For any $\varepsilon > 0$ there exists a measure-preserving homeomorphism $h : X \to X$ such that $h \restriction A = \mathrm{id} \restriction A$, $h[C] \subseteq B$, and $d_{\mathcal{H}}(h, \mathrm{id}) \leq \varepsilon$.

Lemma 9.
Let (X, μ) be a good Cantor measure space, and d be a metric generating the topology of X. Let $B \subseteq X$ be a Borel subset of measure $\mu(B) = \mu(X)$. Let $A \subseteq C$ be two closed nowhere dense subsets in X such that $A \subseteq B$. For any $\varepsilon > 0$ there exists a measure-preserving homeomorphism $h : X \to X$ such that $h \restriction A = \mathrm{id} \restriction A$, $h[C] \subseteq B$, and $d_{\mathcal{H}}(h, \mathrm{id}) \leq \varepsilon$.

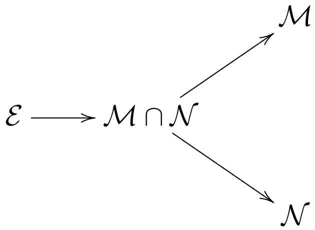
Lemma 10.
Let (X, μ) be a good Cantor measure space and d be a metric generating the topology of X. Let $B \subseteq X$ be a Borel subset of measure $\mu(B) = \mu(X)$. For any $\varepsilon > 0$, homeomorphism $f \in \mathcal{H}_\mu(X)$ and closed nowhere dense subsets $A \subseteq C$ in X with $f[A] \subseteq B$, there exists a homeomorphism $g \in \mathcal{H}_\mu(X)$ such that $g \restriction A = f \restriction A$, $g[C] \subseteq B$ and $d_{\mathcal{H}}(f, g) < \varepsilon$.

Lemma 11.
For any meager F_σ-sets $A, B \subseteq X$ of measure $\mu(A) = \mu(B) = \mu(X)$ in a good Cantor measure space (X, μ) there is a measure-preserving homeomorphism $h \in H_\mu(X)$ such that $h[A] = B$.

Lemma 11.
For any meager F_σ-sets $A, B \subseteq X$ of measure $\mu(A) = \mu(B) = \mu(X)$ in a good Cantor measure space (X, μ) there is a measure-preserving homeomorphism $h \in H_\mu(X)$ such that $h[A] = B$.

Proposition 4.
If $\mathcal{I} \not\subseteq \mathcal{M}$, then $\mathcal{N} \subseteq \mathcal{I}$.

Thank you for your attention!



Taras Banakh, Robert Rałowski, Szymon Żeberski, *Classifying invariant σ-ideals with analytic base on good Cantor measure spaces*, Proc. Amer. Math. Soc., 144 (2016) 837-851.

Open problem.

Classify all nontrivial invariant σ-ideals $\mathcal{I}$ having Borel base for homogenous (X, μ).