Descriptive set theory and geometrical paradoxes II

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Borel circle squaring

Theorem (M.-Unger, 2016)

Tarski's circle squaring problem can be solved using Borel pieces. More generally, suppose $k \geq 1$ and $A, B \subseteq \mathbb{R}^k$ are bounded Borel sets such that $\lambda(A) = \lambda(B) > 0$, $\Delta(\partial A) < k$, and $\Delta(\partial B) < k$. Then A and B are equidecomposable by translations using Borel pieces.

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Fix a sufficiently large d and randomly pick $u_1, \ldots, u_d \in \mathbb{T}^k$. Obtain an action a of $\mathbb{Z}^d$ on $\mathbb{T}^k$ by letting the ith generator of $\mathbb{Z}^d$ act via u_i.

$$(n_1, \ldots, n_d) \cdot x = n_1 u_1 + \ldots + n_d u_d + x$$

Laczkovich shows A and B are a-equidecomposable.

Laczkovich's ideas: Discrepancy theory

If $F \subseteq \mathbb{T}^k$ is finite and $C \subseteq \mathbb{T}^k$ is Lebesgue measurable, then the **discrepancy of F with respect to C is**

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Lemma (Laczkovich 1992 building on Schmidt, Niederreiter-Wills)

For A, B and the action as above, $\exists \epsilon > 0$ and M such that

$$D(R_N \cdot x, A) \leq MN^{-1-\epsilon} \text{ and } D(R_N \cdot x, B) \leq MN^{-1-\epsilon}.$$

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Roughly, every square of side length N in the action contains close to $\lambda(A)N^d$ elements of both A and B.

Flows in graphs

Suppose G is a graph (symmetric irreflexive relation) on a vertex set X. If $f : X \to \mathbb{R}$ is a function, then an f-**flow** of G is a function $\phi \colon G \to \mathbb{R}$ such that

- For every edge $(x, y) \in G$, $\phi(x, y) = -\phi(y, x)$, and
- For every vertex $x \in X$,

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In finite graph theory, flows are usually studied with a single source and sink (e.g. in the max-flow min-cut theorem). For finite graphs, the above type of flow problem is equivalent to one with a single source and sink (by adding a "supersource" and "supersink" to the graph).

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Flows and equidecompositions

For the rest of the proof, let G be the graph with vertex set $\mathbb{T}^k$ where $x, y \in \mathbb{T}^k$ are adjacent if there is $g \in \mathbb{Z}^d$ such that $g \cdot x = y$ where $|g|_\infty = 1$.

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Proposition

A and B are a-equidecomposable with Borel pieces iff there is a bounded Borel integer-valued $\chi_A - \chi_B$-flow of G.

$\rightarrow$: A and B are a-equidecomposable with Borel pieces iff there is Borel bijection $\theta \colon A \to B$ and a finite set $S \subseteq \mathbb{Z}^d$ such that $\forall x \in A \exists g \in S(\theta(x) = g \cdot x)$.

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To construct a flow, for each $x \in A$ add 1 unit of flow to each edge along the lex-least path from x to $\theta(x)$.

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Now construct a Borel bijection from A to B witnessing equidecomposability. Suppose R, S are adjacent tiles. If

$$\sum_{(x,y) \in G \,:\, x \in R \wedge y \in S} \phi(x, y) > 0$$

map this many points of $A \in R$ to points of $B \in S$. If the quantify is negative, map this many points of $B \in R$ to $A \in S$.

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map this many points of $A \in R$ to points of $B \in S$. If the quantify is negative, map this many points of $B \in R$ to $A \in S$. Since ϕ is a $\chi_A - \chi_B$-flow, after doing this the same number of points of A and B remain in each tile. Biject them to finish the construction.

How to construct tilings

An **independent set** in a graph G is a set of vertices where no two are adjacent.

Theorem (Kechris, Solecki, Todorcevic, 1999)

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Let $G^{\leq n}$ be the graph on $\mathbb{T}^k$ where x, y are adjacent if $d_G(x, y) \leq n$. Let C be a Borel maximal independent set for $G^{\leq n}$. Use the element of C as center points for "tiles" of G.

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If we use these center points to make "Voroni cells", the resulting tiling suffices. Gao-Jackson (2015) give a more complicated construction to make rectangular tilings.

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2. We show that given any real-valued Borel f-flow of G, we can find an integer valued Borel f-flow which is "close" to the real-valued one.
3. We finish by using the proposition we've proved above: there's a Borel equidecomposition iff there is a bounded Borel $\chi_A - \chi_B$-flow.

Finding a real-valued bounded flow

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For every $i > 0$, let $\pi_i \colon \mathbb{Z}^d/(2^i\mathbb{Z})^d \to \mathbb{Z}^d/(2^{i-1}\mathbb{Z})^d$ be the canonical homomorphism. This yields the inverse limit

$$\hat{\mathbb{Z}}^d = \varprojlim_{i \geq 0} \mathbb{Z}^d/(2^i\mathbb{Z})^d$$

where elements of $\hat{\mathbb{Z}}^d$ are sequences $(h_0, h_1, \ldots)$ such that $\pi_i(h_i) = h_{i-1}$ for all $i > 0$.

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For each $h \in \hat{\mathbb{Z}}^d$ and $x \in \mathbb{T}^k$, we give an explicit construction $\phi_{x,h}$ of a flow of the connected component of x. However, we cannot pick a single x in each orbit to be a "starting point" for this construction (since this would be a nonmeasurable Vitali set).

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The construction is such that if $g \in \mathbb{Z}^d$, then $\phi_{x,h} = \phi_{g\cdot x,-g+h}$. Hence, the average value of this construction is invariant of our starting point ($h \mapsto -g + h$ is measure preserving):

$$\int_h \phi_{x,h} = \int_h \phi_{g\cdot x,-g+h} = \int_h \phi_{g\cdot x,h}$$

In the last lecture, we'll discuss how to turn a real-valued flow of G into an integer-valued flow. This step uses:

- the Ford-Fulkerson algorithm in finite combinatorics.
- work of A. Timár on boundaries of finite sets in $\mathbb{Z}^d$.
- very recent work of Gao, Jackson, Krohne and Seward on hyperfiniteness of free Borel actions of $\mathbb{Z}^d$.