An Introduction to Hyperstationary Sets

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Outline

Provability Logic

Provability Logic is the logic in the language of propositional logic with an additional modal operator $\Box$.

Axioms:

1. Boolean tautologies.
2. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
3. $\Box(\Box\varphi \to \varphi) \to \Box\varphi$

Rules:

1. $\vdash \varphi, \vdash \varphi \to \psi \Rightarrow \vdash \psi$ (Modus Ponens)
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The Logic $\mathbf{GLP}_\omega$

One may introduce additional modal operators $[n]$, for each $n < \omega$. The corresponding dual operators $\neg[n]\neg$ are denoted by $\langle n \rangle$. The logic system $\mathbf{GLP}_\omega$ (Japaridze, 1986) has the following axioms and rules:

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More generally, for any ordinal $\xi \geq 2$, one considers the language of propositional logic with additional modal operators $[\alpha]$, for each $\alpha < \xi$. The corresponding dual operators $\neg[\alpha]\neg$ being denoted by $\langle\alpha\rangle$. The logic system $\mathbf{GLP}_\xi$ has the following axioms and rules:

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Topological semantics

People have been interested in proving completeness for $\mathbf{GLP}_\xi$, with respect to some natural semantics.

Problem: Kripke-style semantics do not work!

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Topological semantics

Thus, one considers polytopological spaces $(X, (\tau_\alpha)_{\alpha<\xi})$.

A valuation on X is a map $v : Form \to \mathcal{P}(X)$ such that:

1. $v(\neg\varphi) = X - v(\varphi)$
2. $v(\varphi \wedge \psi) = v(\varphi) \cap v(\psi)$
3. $v(\langle\alpha\rangle\varphi) = D_\alpha(v(\varphi))$, for all $\alpha < \xi$, where $D_\alpha : \mathcal{P}(X) \to \mathcal{P}(X)$ is the derived set operator for τ_α (i.e., $D_\alpha(A)$ is the set of limit points of A in the τ_α topology).
   Hence, $v([\alpha]\varphi) = X - D_\alpha(X - v(\varphi)) =$ the τ_α-interior of $v(\varphi)$, for all $\alpha < \xi$.

A formula is valid in X if $v(\varphi) = X$, for every valuation v on X.

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For the $\mathbf{GLP}_\xi$ axioms to be valid in $(X, (\tau_\alpha)_{\alpha<\xi})$, the topologies τ_α have to satisfy:

1. τ_α is scattered, all $\alpha < \xi$.
2. $\tau_\beta \subseteq \tau_\alpha$, for all $\beta \leq \alpha < \xi$.
3. $D_\alpha(A)$ is an open set in $\tau_{\alpha+1}$, for all $A \subseteq X$.

   Moreover, for $\mathbf{GLP}_\xi$ to be complete, one must also have:

4. The τ_α are non-trivial (i.e., non discrete).

So, one doesn't have much choice on how to define the τ_α: One fixes a scattered topology τ_0 on X, and the other topologies are determined by the D_α operators. One only needs to make sure the τ_α are non-trivial.

Such polytopological spaces are called general **GLP**-spaces.

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Ordinal **GLP**-spaces

Fix some limit ordinal δ (we also allow $\delta = OR$).

Recall that the order topology on δ (a. k. a. the interval topology) is the topology τ_0 generated by the set $\mathcal{B}_0$ consisting of $\{0\}$ and the intervals (α, β).

τ_0 is a Hausdorff scattered topology in which 0 and all successor ordinals are isolated points, and the accumulation points are precisely the limit ordinals.

Now define a continuous sequence of derived topologies

$$\tau_0 \subseteq \tau_1 \subseteq \ldots \subseteq \tau_\xi \subseteq \ldots$$

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Derived Topologies

Given τ_ξ, let $D_\xi : \mathcal{P}(\delta) \to \mathcal{P}(\delta)$ be the Cantor derivative operator:

$$D_\xi(A) := \{\alpha \in \delta : \alpha \text{ is a limit point of } A \text{ in the } \tau_\xi \text{ topology}\}.$$

Note that $D_\xi(A)$ is a closed set in the τ_ξ topology.

Then let $\tau_{\xi+1}$ be the topology generated by the set

$$\mathcal{B}_{\xi+1} := \mathcal{B}_\xi \cup \{D_\xi(A) : A \subseteq \delta\}.$$

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Derived Topologies

Notice that if the cofinality of α is uncountable and $\alpha \in D_0(A)$, then $D_0(A) \cap \alpha$ is a club subset of α.

The set $\mathcal{B}_1 := \mathcal{B}_0 \cup \{D_0(A) : A \subseteq \delta\}$ is a base for the topology τ_1 on OR, known as the club topology.

Note that the non-isolated points are exactly the ordinals of uncountable cofinality.

Fact

For every set of ordinals A,

$$D_1(A) = \{\alpha : A \cap \alpha \text{ is stationary in } \alpha\}.$$

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The next topology, τ_2, is generated by the set

$$\mathcal{B}_2 := \mathcal{B}_1 \cup \{D_1(A) : A \subseteq OR\}.$$

If some stationary subset S of α does not reflect (i.e., $D_1(S) = \{\alpha\}$), then α is an isolated point of τ_2. Thus, every non-isolated point α has to reflect all stationary sets.

Further, if some stationary subsets S, T of α do not simultaneously reflect (i.e., $D_1(S) \cap D_1(T) = \{\alpha\}$), then α is an isolated point of τ_2. Thus, every non-isolated point has to reflect simultaneously all pairs of stationary sets.

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Stationary reflection

An ordinal α of uncountable cofinality reflects stationary sets if for every stationary $A \subseteq \alpha$ there exists $\beta < \alpha$ such that $A \cap \beta$ is stationary in β.

Let us say that an ordinal α of uncountable cofinality is simultaneoulsy-stationary-reflecting if every pair A, B of stationary subsets of α simultaneously reflect, that is, there exists $\beta < \alpha$ such that $A \cap \beta$ and $B \cap \beta$ are both stationary in β.

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Jensen's Theorem

It is easy to see that every weakly-compact cardinal (i.e., Π^1_1-indescribable) is simultaneously-stationary-reflecting.

Theorem (Jensen)

In the constructible universe L a regular cardinal κ reflects stationary sets if and only if it is Π^1_1-indescribable, hence if and only if it is simultaneously-stationary-reflecting.[a]

[a] R. Jensen, The fine structure of the constructible hierarchy. Annals of Math. Logic 4 (1972)

Thus, in L, the non-isolated points of the topology τ_2 are precisely the ordinals whose cofinality is a weakly-compact cardinal.

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ξ-stationary sets

Definition

We say that $A \subseteq \delta$ is 0-stationary in α, α a limit ordinal, if and only if $A \cap \alpha$ is unbounded in α.
For $\xi > 0$, we say that A is ξ-stationary in α if and only if for every $\zeta < \xi$, every subset S of α that is ζ-stationary in α ζ-reflects to some $\beta \in A$, i.e., $S \cap \beta$ is ζ-stationary in β.

Note:

1. A is 1-stationary in $\alpha \Leftrightarrow A$ is stationary in α, in the usual sense.
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ξ-stationary reflection

Definition

We say that $A \subseteq \delta$ is 0-simultaneously-stationary in α (0-s-stationary in α, for short) if and only if $A \cap \alpha$ is unbounded in α.

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Lecture II

Recall from Lecture I

We are looking at ordinal **GLP**-spaces, i.e., polytopological spaces of the form $(\delta, (\tau_\zeta)_{\zeta<\xi})$, where τ_0 is the interval topology and $\tau_{\zeta+1}$ is generated by τ_ζ together with the sets

$$D_\zeta(A) := \{\alpha : \alpha \text{ is a } \tau_\zeta \text{ limit point of } A\}$$

all $A \subseteq \delta$.

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Proposition

α is not isolated in the τ_2 topology if and only if α is 2-s-stationary

Proof.

If α is not 2-s-stationary, there are stationary $A, B \subseteq \alpha$ such that $D_1(A) \cap D_1(B) = \{\alpha\}$, hence α is isolated.
Now suppose α is 2-s-stat. and $\alpha \in U = C \cap D_1(A_0) \cap \ldots \cap D_1(A_{n-1})$, where $C \subseteq \alpha$ is club. We claim that U contains some ordinal other than α. It is enough to show that $D_1(A_0) \cap \ldots \cap D_1(A_{n-1})$ is stationary.
Suppose first that $n = 2$. Fix any club $C' \subseteq \alpha$. The sets $C' \cap A_0$ and $C' \cap A_1$ are stationary in α, and therefore they simultaneously reflect at some $\beta < \alpha$. Thus $\beta \in C' \cap D_1(A_0) \cap D_1(A_1)$.
Now, assume it holds for n and let us show it holds for $n+1$. Fix a club $C' \subseteq \alpha$. By the ind. hyp., $C' \cap D_1(A_0) \cap \ldots \cap D_1(A_{n-1})$ is stationary. So, since the proposition holds for $n = 2$, the set $D_1(C' \cap D_1(A_0) \cap \ldots \cap D_1(A_{n-1})) \cap D_1(A_n)$ is also stationary. But clearly $D_1(C' \cap D_1(A_0) \cap \ldots \cap D_1(A_{n-1})) \cap D_1(A_n) \subseteq C' \cap D_1(A_0) \cap \ldots \cap D_1(A_n)$. □

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□

A similar argument, relativized to any set A yields:

Proposition

$D_2(A) = \{\alpha : A \cap \alpha \text{ is 2-s-stationary in } \alpha\}$.

The τ_ξ topology

In order to analyse the topologies τ_ξ, for $\xi \geq 3$, note first the following general facts:

1. For every $\xi' < \xi$ and every $A, B \subseteq \delta$,

$$D_{\xi'}(A) \cap D_\xi(B) = D_\xi(D_{\xi'}(A) \cap B).$$

2. For every ordinal ξ, the sets of the form

$$I \cap D_{\xi'}(A_0) \cap \ldots \cap D_{\xi'}(A_{n-1})$$

where $I \in \mathcal{B}_0$, $n < \omega$, $\xi' < \xi$, and $A_i \subseteq \delta$, all $i < n$, form a base for τ_ξ.

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Characterizing non-isolated points

Theorem

1. *For every* ξ,

$$D_\xi(A) = \{\alpha : A \text{ is } \xi\text{-s-stationary in } \alpha\}.^a$$

2. *For every* ξ *and* α, A *is* $\xi + 1$*-s-stationary in* α *if and only if* $A \cap D_\zeta(S) \cap D_\zeta(T) \cap \alpha \neq \emptyset$ *(equivalently, if and only if* $A \cap D_\zeta(S) \cap D_\zeta(T)$ *is* ζ*-s-stationary in* α*) for every* $\zeta \leq \xi$ *and every pair* S, T *of subsets of* α *that are* ζ*-s-stationary in* α.
3. *For every* ξ *and* α, *if* A *is* ξ*-s-stationary in* α *and* A_i *is* ζ_i*-s-stationary in* α *for some* $\zeta_i < \xi$, *all* $i < n$, *then* $A \cap D_{\zeta_0}(A_0) \cap \ldots \cap D_{\zeta_{n-1}}(A_{n-1})$ *is* ξ*-s-stationary in* α.

[a] For $\xi < \omega$, this is due independently to L. Beklemishev (Unpublished).

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3. *For every ξ and α, if A is ξ-s-stationary in α and A_i is ζ_i-s-stationary in α for some $\zeta_i < \xi$, all $i < n$, then $A \cap D_{\zeta_0}(A_0) \cap \ldots \cap D_{\zeta_{n-1}}(A_{n-1})$ is ξ-s-stationary in α.*

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Taking $A = \delta$ in (1) above, we obtain the following

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For every ξ, an ordinal $\alpha < \delta$ is not isolated in the τ_ξ topology if and only if α is ξ-s-stationary.

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The ideal of non-ξ-s-stationary sets

For each limit ordinal α and each ξ, let NS_α^ξ be the set of non-ξ-s-stationary subsets of α.

Thus, if α has uncountable cofinality, NS_α^1 is the ideal of non-stationary subsets of α and $(NS_\alpha^1)^*$ is the club filter over α.

Notice that $\zeta \leq \xi$ implies $NS_\alpha^\zeta \subseteq NS_\alpha^\xi$ and $(NS_\alpha^\zeta)^* \subseteq (NS_\alpha^\xi)^*$.

Also note that $A \subseteq \alpha$ belongs to $(NS_\alpha^\xi)^*$ if and only if for some $\zeta < \xi$ and some ζ-s-stationary sets $S, T \subseteq \alpha$, the set $D_\zeta(S) \cap D_\zeta(T) \cap \alpha$ is contained in A. In particular, if $S \subseteq \alpha$ is ζ-s-stationary, with $\zeta < \xi$, then $D_\zeta(S) \cap \alpha \in (NS_\alpha^\xi)^*$.

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For each limit ordinal α and each ξ, let NS^{ξ}_{α} be the set of non-ξ-s-stationary subsets of α.

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Theorem

For every ξ, a limit ordinal α is ξ-s-stationary if and only if NS^{ξ}_{α} is a proper ideal, hence if and only if $(NS^{\xi}_{\alpha})^{*}$ is a proper filter.

Proof.

Assume α is ξ-s-stationary (hence $\alpha \notin NS^{\xi}_{\alpha}$) and let us show that NS^{ξ}_{α} is an ideal. For $\xi = 0$ this is clear. So, suppose $\xi > 0$ and $A, B \in NS^{\xi}_{\alpha}$. There exist $\zeta_A, \zeta_B < \xi$, and there exist sets S_A, $T_A \subseteq \alpha$ that are ζ_A-s-stationary in α, and sets S_B, $T_B \subseteq \alpha$ that are ζ_B-s-stationary in α, such that $D_{\zeta_A}(S_A) \cap D_{\zeta_A}(T_A) \cap A = D_{\zeta_B}(S_B) \cap D_{\zeta_b}(T_B) \cap B = \emptyset$. Hence,

$$(D_{\zeta_A}(S_A) \cap D_{\zeta_A}(T_A) \cap D_{\zeta_B}(S_B) \cap D_{\zeta_B}(T_B)) \cap (A \cup B) = \emptyset.$$

The set $X := D_{\zeta_A}(S_A) \cap D_{\zeta_A}(T_A) \cap D_{\zeta_B}(S_B) \cap D_{\zeta_B}(T_B)$ is $max\{\zeta_A, \zeta_B\}$-s-stationary in α. Now notice that

$$D_{max\{\zeta_A,\zeta_B\}}(X) \subseteq X$$

and so we have

$$D_{max\{\zeta_A,\zeta_B\}}(X) \cap \alpha \cap (A \cup B) = \emptyset$$

which witnesses that $A \cup B \in NS^{\xi}_{\alpha}$. □

Continued.

For the converse, assume NS^{ξ}_{α} is a proper ideal.
Take any A and B ζ-s-stationary subsets of α, for some $\zeta < \xi$. Then $D_{\zeta}(A) \cap \alpha$ and $D_{\zeta}(B) \cap \alpha$ are in $(NS^{\xi}_{\alpha})^*$. Moreover, if $S, T \subseteq \alpha$ are any ζ'-s-stationary sets, for some $\zeta' < \xi$, then also $D_{\zeta'}(S) \cap \alpha$ and $D_{\zeta'}(T) \cap \alpha$ belong to $(NS^{\xi}_{\alpha})^*$. Hence, since $(NS^{\xi}_{\alpha})^*$ is a filter,

$$D_{\zeta}(A) \cap D_{\zeta}(B) \cap D_{\zeta'}(S) \cap D_{\zeta'}(T) \cap \alpha \in (NS^{\xi}_{\alpha})^*$$

which implies, since $(NS^{\xi}_{\alpha})^*$ is proper, that $D_{\zeta}(A) \cap D_{\zeta}(B) \cap D_{\zeta'}(S) \cap D_{\zeta'}(T) \cap \alpha \neq \emptyset$. This shows that $D_{\zeta}(A) \cap D_{\zeta}(B)$ is ξ-s-stationary in α. Since A and B were arbitrary, this implies α is ξ-s-stationary. □

Summary

The following are equivalent for every limit ordinal α and every $\xi > 0$:

1. α is a non-isolated point in the τ_ξ topology.
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Indescribable cardinals

Recall that a formula of second-order logic is Σ^1_0 (or Π^1_0) if it does not have quantifiers of second order, but it may have any finite number of first-order quantifiers and free first-order and second-order variables.

Definition

For ξ any ordinal, we say that a formula is $\Sigma^1_{\xi+1}$ if it is of the form

$$\exists X_0, \ldots, X_k \varphi(X_0, \ldots, X_k)$$

where $\varphi(X_0, \ldots, X_k)$ is Π^1_ξ.
And a formula is $\Pi^1_{\xi+1}$ if it is of the form

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Definition

If ξ is a limit ordinal, then we say that a formula is Π^1_ξ if it is of the form

$$\bigwedge_{\zeta<\xi} \varphi_\zeta$$

where φ_ζ is Π^1_ζ, all $\zeta < \xi$, and it has only finitely-many free second-order variables. And we say that a formula is Σ^1_ξ if it is of the form

$$\bigvee_{\zeta<\xi} \varphi_\zeta$$

where φ_ζ is Σ^1_ζ, all $\zeta < \xi$, and it has only finitely-many free second-order variables.

Definition

A cardinal κ is Π^1_ξ-*indescribable* if for all subsets $A \subseteq V_\kappa$ and every Π^1_ξ sentence φ, if

$$\langle V_\kappa, \in, A \rangle \models \varphi$$

then there is some $\lambda < \kappa$ such that

$$\langle V_\lambda, \in, A \cap V_\lambda \rangle \models \varphi.$$

Theorem

Every Π^1_ξ-indescribable cardinal is $(\xi+1)$-s-stationary. Hence, if ξ is a limit ordinal and a cardinal κ is Π^1_ζ-indescribable for all $\zeta<\xi$, then κ is ξ-s-stationary.

Proof.

Let κ be an infinite cardinal. Clearly, the fact that a set $A \subseteq \kappa$ is 0-s-stationary (i.e., unbounded) in κ can be expressed as a Π^1_0 sentence $\varphi_0(A)$ over $\langle V_\kappa, \in, A\rangle$. Inductively, for every $\xi > 0$, the fact that a set $A \subseteq \kappa$ is ξ-s-stationary in κ can be expressed by a Π^1_ξ sentence φ_ξ over $\langle V_\kappa, \in, A\rangle$. Namely,

$$\bigwedge_{\zeta<\xi} (A \text{ is } \zeta\text{-s-stationary})$$

in the case ξ is a limit ordinal, and by the sentence

$$\bigwedge_{\zeta<\xi-1} (A \text{ is } \zeta\text{-s-stationary}) \wedge$$

$$\forall S, T (S, T \text{ are } (\xi-1)\text{-s-stationary in } \kappa \rightarrow$$

$$\exists \beta \in A (S \text{ and } T \text{ are } (\xi-1)\text{-s-stationary in } \beta))$$

which is easily seen to be equivalent to a Π^1_ξ sentence, in the case ξ is a successor ordinal. □

Continued.

Now suppose κ is Π^1_ξ-indescribable, and suppose that A and B are ζ-s-stationary subsets of κ, for some $\zeta \leq \xi$. Thus,

$$\langle V_\kappa, \in, A, B \rangle \models \varphi_\zeta[A] \wedge \varphi_\zeta[B].$$

By the Π^1_ζ-indescribability of κ there exists $\beta < \kappa$ such that

$$\langle V_\beta, \in, A \cap \beta, B \cap \beta \rangle \models \varphi_\zeta[A \cap \beta] \wedge \varphi_\zeta[B \cap \beta]$$

which implies that A and B are ζ-s-stationary in β. Hence κ is $(\xi + 1)$-s-reflecting. □

Reflection and indescribability in L

Theorem (J.B.-M. Magidor-H. Sakai, 2013; J.B., 2015)

Assume $V = L$. For every $\xi > 0$, a regular cardinal is $(\xi + 1)$-stationary if and only if it is Π^1_ξ-indescribable, hence if and only if it is $(\xi + 1)$-s-stationary.[a][b]

[a] *Reflection and indescribability in the constructible universe*. Israel J. of Math. Vol. 208, Issue 1 (2015)

[b] *Derived topologies on ordinals and stationary reflection*. Preprint (2015)

The proof actually shows the following:

Theorem

Assume $V = L$. *Suppose* $\xi > 0$ *and* κ *is a regular* $(\xi + 1)$*-stationary cardinal. Then for every* $A \subseteq \kappa$ *and every* Π^1_ξ *sentence* Ψ, *if* $\langle L_\kappa, \in, A\rangle \models \Psi$, *then there exists a* ξ*-stationary* $S \subseteq \kappa$ *such that* Ψ *reflects to every ordinal* λ *on which* S *is* ξ*-stationary.*

Theorem

CON($\exists\kappa < \lambda$ (κ *is* Π^1_ξ*-indescribable* $\wedge$ λ *is inaccessible*)) *implies* *CON*($\tau_{\xi+1}$ *is non-discrete* $\wedge$ $\tau_{\xi+2}$ *is discrete*).

Proof.

Let κ be Π^1_ξ-indescribable, and let $\lambda > \kappa$ be inaccessible. In L, κ is Π^1_ξ-indescribable and λ is inaccessible. So, in L, let κ_0 be the least Π^1_ξ-indescribable cardinal, and let λ_0 be the least inaccessible cardinal above κ_0. Then L_{λ_0} is a model of ZFC in which κ_0 is Π^1_ξ-indescribable and no regular cardinal greater than κ_0 is 2-stationary. The reason is that if α is a regular cardinal greater than κ_0, then $\alpha = \beta^+$, for some cardinal β. And since Jensen's principle $\Box_\beta$ holds, there exists a stationary subset of α that does not reflect. $\square$

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Lecture III

Recall from Lecture II

If $V = L$, then the following are equivalent for every regular cardinal κ and $\xi > 0$:

1. κ if $(\xi + 1)$-stationary.
2. κ is $(\xi + 1)$-s-stationary.
3. κ is Π^1_ξ-indescribable.

Hence, for every limit ordinal ξ, a regular cardinal is ξ-stationary if and only if it is ξ-s-stationary, and if and only if it is Π^1_ζ-indescribable for every $\zeta < \xi$.

Question

What is the consistency strength of ξ-stationarity? And of ξ-s-stationarity?

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The consistency strength of 2-stationarity

Let us write:

$$d_\xi(A) := \{\alpha : A \cap \alpha \text{ is } \xi\text{-stationary in } \alpha\}$$

Definition (A. H. Mekler-S. Shelah, 1989)

A regular uncountable cardinal κ is a reflection cardinal if there exists a reflection ideal on κ, i.e., a proper, normal, and κ-complete ideal $\mathcal{I}$ on κ such that for every $X \subseteq \kappa$,

$$X \in \mathcal{I}^+ \quad \Rightarrow \quad d_1(X) \in \mathcal{I}^+.$$

Note: if κ is 2-stationary, then NS_κ is the smallest such ideal.
κ is weakly compact $\Rightarrow$ many reflection cardinals below κ.

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If κ is a reflection cardinal in L, then in some generic extension of L that preserves cardinals, κ is 2-stationary. (In fact, the set Reg $\cap \kappa$ of regular cardinals below κ is 2-stationary.)

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The following are equiconsistent:

1. *There exists a reflection cardinal.*
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On the consistency strength of 2-stationarity

Definition

A regular cardinal κ is greatly Mahlo if there exists a proper, normal, and κ-complete ideal $\mathcal{I}$ on κ such that $Reg \cap \kappa \in \mathcal{I}^*$, and for every $X \subseteq \kappa$,

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Theorem (A. H. Mekler-S. Shelah, 1989)

If $V = L$ and κ is at most the first greatly-Mahlo cardinal, then κ is not a reflection cardinal.

Thus, in L, the first reflection cardinal is strictly between the first greatly-Mahlo and the first weakly-compact.

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On the consistency strength of ξ-stationarity

We would like to prove analogous results for ξ-stationay sets. So, let's define:

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For $\xi > 0$, a regular uncountable cardinal κ is an ξ-reflection cardinal if there exists a ξ-reflection ideal on κ, i.e., a proper, normal, and κ-complete ideal $\mathcal{I}$ on κ such that for every $X \subseteq \kappa$,

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Note: κ is 2-stationary if and only if NS_κ is a 1-reflection ideal. Thus, every 2-stationary regular cardinal is a 1-reflection cardinal.

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Proposition

Every Π^1_ξ-indescribable cardinal is a $(\xi+1)$-reflection cardinal.

Proof.

If κ is Π^1_ξ-indescribable, then $NS^{\xi+1}_\kappa$ is a $(\xi+1)$-reflection ideal. The point is that if κ is Π^1_ξ-indescribable, then $(NS^{\xi+1}_\kappa)^*$ is contained in the $(\xi+1)$-indescribable filter, and hence it is normal. □

However,

Proposition

For every $\xi > 0$, the fact that κ is a ξ-reflection cardinal is Π^1_1 expressible over the structure $\langle V_\kappa, \in, \xi, \kappa\rangle$. Hence, if κ is a ξ-reflection cardinal and is weakly compact, then the set of ξ-reflection cardinals smaller than κ belongs to the weakly compact filter.

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On the consistency strength of ξ-stationarity

Theorem (J.B., M. Magidor, and S. Mancilla, 2015)

If κ is a ξ-reflection cardinal in L, then in some generic extension of L that preserves cardinals, κ is $(\xi+1)$-stationary.
(In fact, the set Reg $\cap\, \kappa$ of regular cardinals below κ is $(\xi+1)$-stationary).

Problem

Suppose S is a subset of κ that does not 2-reflect, i.e., $d_2(S) = \emptyset$. Then $T := S \cup \{\alpha < \kappa : cof(\alpha) = \omega\}$ does not 2-reflect either: for if $\alpha \in d_2(T)$, then since $\alpha \notin d_2(S)$ there exists $X \subseteq \alpha$ i-stationary, some $i < 2$, such that $d_i(X) \cap S \cap \alpha = \emptyset$. If $i = 0$, then $d_i(X) \cap \alpha$ is a club subset of α disjoint from S, and therefore $d_i(X) \cap T \cap \alpha$ is a 2-stationary subset of α contained in $\{\beta < \alpha : cof(\beta) = \omega\}$, which is impossible. But if $i = 1$, then $d_i(X) \cap T \cap \alpha = d_i(X) \cap S \cap \alpha = \emptyset$, contradicting $\alpha \in d_2(T)$.

Now, if we shoot a club through the complement of T, then in $V[G]$ the club contains ordinals of cofinality ω but whose cofinality in V is uncountable. Hence cardinals are collapsed.

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Definition

For κ an uncountable regular cardinal, $S \subseteq \kappa$, and $\xi > 0$, let $\mathbb{D}_{\xi,S}$ be the forcing notion whose conditions are functions

$$p : \delta + 1 \to \{0, 1\}$$

where $\delta < \kappa$ and $p^{-1}[\{1\}]$ is not ξ-stationary in α for every $\alpha \in S$, i.e., $d_\xi(p^{-1}[\{1\}]) \subseteq \kappa \setminus S$. The ordering is by end-extension, i.e., $p \leq q$ if and only if p is an end-extension of q.

Lemma

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Lemma

Suppose that H is $\mathbb{D}_{\xi,S}$-generic over V and let

$$X_H := \bigcup\{p^{-1}[\{1\}] : p \in H\}.$$

Then X_H is a stationary subset of κ and $d_\xi(X_H) \cap \kappa \subseteq \kappa \setminus S$.

The iteration

We do an iteration $\mathbb{P}$, of length κ^+, with support of size $< \kappa$, and such that at every successor stage α, if the subset S of κ given by the bookkeeping function is a stationary set that does not reflect, then the forcing $\dot{\mathbb{Q}}_\alpha$ shoots a club through the complement of S; and if S is a stationary set such that $d_\zeta(S) \neq \emptyset$ but $d_{\zeta+1}(S) = \emptyset$, some $0 < \zeta \leq \xi$, then $\dot{\mathbb{Q}}_\alpha$ adds a set of the form $d_\zeta(X)$, with X stationary, through the complement of S. Moreover, we destroy at later stages of the iteration all potential counterexamples to X being ζ-stationary

On the consistency strength of n-stationarity

Definition

A regular cardinal κ is ξ-greatly Mahlo if there exists a proper, normal, and κ-complete ideal $\mathcal{I}$ on κ such that $Reg \cap \kappa \in \mathcal{I}^*$, and for every $X \subseteq \kappa$,

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Theorem (J.B. and S. Mancilla, 2014)

In L, if κ is at most the first ξ-greatly-Mahlo cardinal, then κ is not an ξ-reflection cardinal.

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Conclusion

Corollary

The consistency strength of the existence of an $(\xi + 1)$-stationary cardinal is strictly between the existence of a ξ-greatly-Mahlo cardinal and the existence of a Π^1_ξ-indescribable cardinal.

On the consistency strength of ξ-s-stationarity.

Theorem (Magidor)

The following are equiconsistent:

1. *There exists a 2-s-stationary cardinal (i.e., a cardinal that reflects simultaneously pairs of stationary sets).*
2. *There exists a weakly-compact cardinal.*[a]

[a] M. Magidor, On reflecting stationary sets. JSL 47 (1982)

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The following should be equiconsistent for every $\xi > 0$:

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The **GLP** completeness problem

In order to solve the **GLP** completeness problem under ordinal topological semantics it only remains to prove the following:

Theorem (?)

Assume whatever you need (e.g., large cardinals, global square, ...). For $\xi > 1$ and some κ, for every finite rooted tree $\langle T, \leq_T \rangle$, there exists a function $S : T \to \mathcal{P}(\kappa) \setminus \{\emptyset\}$ such that

1. *$\{S_x : x \in T\}$ is pairwise disjoint.*
2. *If $x <_T y$ and $\alpha \in S_x$, then $S_y \cap \alpha \in (NS^{\xi}_{\alpha})^{+}$.*
3. *For every $x \in T$, if $\alpha \in S_x$, then $(\bigcup_{x <_T y} S_y) \cap \alpha \in (NS^{\xi}_{\alpha})^{*}$.*

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Open for exploration

1. Develop the theory of hyperstationary sets for $\mathcal{P}_\kappa(\lambda)$. What are the large cardinals involved?
2. Define the hyperstationary version of Woodin's stationary tower and study its properties.
3. Characterize the non-isolated points of general **GLP**-spaces. What are the large cardinals involved?
4. What is the notion of almost-freeness for abelian groups that corresponds (i.e., is equiconsistent) to ξ-stationarity?
5. Take any result about stationary sets and prove it or disprove it for hyperstationary sets (assuming appropriate large cardinals).

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