Lecture II

Recall from Lecture I

We are looking at ordinal **GLP**-spaces, i.e., polytopological spaces of the form $(\delta, (\tau_\zeta)_{\zeta<\xi})$, where τ_0 is the interval topology and $\tau_{\zeta+1}$ is generated by τ_ζ together with the sets

$$D_\zeta(A) := \{\alpha : \alpha \text{ is a } \tau_\zeta \text{ limit point of } A\}$$

all $A \subseteq \delta$.

τ_1 is the club topology. The non-isolated points are those α with uncountable cofinality.

We observed that $D_1(A) = \{\alpha : A \cap \alpha \text{ is stationary in } \alpha\}$.

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Recall also the following definition

Definition

We say that $A \subseteq \delta$ is 0-simultaneously-stationary in α (0-s-stationary in α, for short) if and only if $A \cap \alpha$ is unbounded in α.

For $\xi > 0$, we say that $A \subseteq \delta$ is ξ-simultaneously-stationary in α (ξ-s-stationary in α, for short) if and only for every $\zeta < \xi$, every pair of ζ-s-stationary subsets $B, C \subseteq \alpha$ simultaneously ζ-s-reflect at some $\beta \in A$, i.e., $B \cap \beta$ and $C \cap \beta$ are ζ-s-stationary in β.

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A is 2-s-stationary in $\alpha \Leftrightarrow$ every pair of stationary subsets of α simultaneously reflect to some $\beta \in A$.

Proposition

α is not isolated in the τ_2 topology if and only if α is 2-s-stationary

Proof.

If α is not 2-s-stationary, there are stationary $A, B \subseteq \alpha$ such that $D_1(A) \cap D_1(B) = \{\alpha\}$, hence α is isolated.
Now suppose α is 2-s-stat. and $\alpha \in U = C \cap D_1(A_0) \cap \ldots \cap D_1(A_{n-1})$, where $C \subseteq \alpha$ is club. We claim that U contains some ordinal other than α. It is enough to show that $D_1(A_0) \cap \ldots \cap D_1(A_{n-1})$ is stationary.
Suppose first that $n = 2$. Fix any club $C' \subseteq \alpha$. The sets $C' \cap A_0$ and $C' \cap A_1$ are stationary in α, and therefore they simultaneously reflect at some $\beta < \alpha$. Thus $\beta \in C' \cap D_1(A_0) \cap D_1(A_1)$.
Now, assume it holds for n and let us show it holds for $n+1$. Fix a club $C' \subseteq \alpha$. By the ind. hyp., $C' \cap D_1(A_0) \cap \ldots \cap D_1(A_{n-1})$ is stationary. So, since the proposition holds for $n = 2$, the set $D_1(C' \cap D_1(A_0) \cap \ldots \cap D_1(A_{n-1})) \cap D_1(A_n)$ is also stationary. But clearly $D_1(C' \cap D_1(A_0) \cap \ldots \cap D_1(A_{n-1})) \cap D_1(A_n) \subseteq C' \cap D_1(A_0) \cap \ldots \cap D_1(A_n)$.

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□

A similar argument, relativized to any set A yields:

Proposition

$D_2(A) = \{\alpha : A \cap \alpha$ *is* 2*-s-stationary in* $\alpha\}$.

The τ_ξ topology

In order to analyse the topologies τ_ξ, for $\xi \geq 3$, note first the following general facts:

1. For every $\xi' < \xi$ and every $A, B \subseteq \delta$,

$$D_{\xi'}(A) \cap D_\xi(B) = D_\xi(D_{\xi'}(A) \cap B).$$

2. For every ordinal ξ, the sets of the form

$$I \cap D_{\xi'}(A_0) \cap \ldots \cap D_{\xi'}(A_{n-1})$$

where $I \in \mathcal{B}_0$, $n < \omega$, $\xi' < \xi$, and $A_i \subseteq \delta$, all $i < n$, form a base for τ_ξ.

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Characterizing non-isolated points

Theorem

1. *For every* ξ*,*

$$D_\xi(A) = \{\alpha : A \text{ is } \xi\text{-s-stationary in } \alpha\}.^a$$

2. *For every* ξ *and* α*,* A *is* $\xi + 1$*-s-stationary in* α *if and only if* $A \cap D_\zeta(S) \cap D_\zeta(T) \cap \alpha \neq \emptyset$ *(equivalently, if and only if* $A \cap D_\zeta(S) \cap D_\zeta(T)$ *is* ζ*-s-stationary in* α*) for every* $\zeta \leq \xi$ *and every pair* S*,* T *of subsets of* α *that are* ζ*-s-stationary in* α*.*
3. *For every* ξ *and* α*, if* A *is* ξ*-s-stationary in* α *and* A_i *is* ζ_i*-s-stationary in* α *for some* $\zeta_i < \xi$*, all* $i < n$*, then* $A \cap D_{\zeta_0}(A_0) \cap \ldots \cap D_{\zeta_{n-1}}(A_{n-1})$ *is* ξ*-s-stationary in* α*.*

[a] For $\xi < \omega$, this is due independently to L. Beklemishev (Unpublished).

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3. *For every* ξ *and* α, *if* A *is* ξ*-s-stationary in* α *and* A_i *is* ζ_i*-s-stationary in* α *for some* $\zeta_i < \xi$, *all* $i < n$, *then* $A \cap D_{\zeta_0}(A_0) \cap \ldots \cap D_{\zeta_{n-1}}(A_{n-1})$ *is* ξ*-s-stationary in* α.

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Taking $A = \delta$ in (1) above, we obtain the following

Corollary

For every ξ, an ordinal $\alpha < \delta$ is not isolated in the τ_ξ topology if and only if α is ξ-s-stationary.

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The ideal of non-ξ-s-stationary sets

For each limit ordinal α and each ξ, let NS^{ξ}_{α} be the set of non-ξ-s-stationary subsets of α.

Thus, if α has uncountable cofinality, NS^{1}_{α} is the ideal of non-stationary subsets of α and $(NS^{1}_{\alpha})^{*}$ is the club filter over α.

Notice that $\zeta \leq \xi$ implies $NS^{\zeta}_{\alpha} \subseteq NS^{\xi}_{\alpha}$ and $(NS^{\zeta}_{\alpha})^{*} \subseteq (NS^{\xi}_{\alpha})^{*}$.

Also note that $A \subseteq \alpha$ belongs to $(NS^{\xi}_{\alpha})^{*}$ if and only if for some $\zeta < \xi$ and some ζ-s-stationary sets $S, T \subseteq \alpha$, the set $D_{\zeta}(S) \cap D_{\zeta}(T) \cap \alpha$ is contained in A. In particular, if $S \subseteq \alpha$ is ζ-s-stationary, with $\zeta < \xi$, then $D_{\zeta}(S) \cap \alpha \in (NS^{\xi}_{\alpha})^{*}$.

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The ideal of non-ξ-s-stationary sets

For each limit ordinal α and each ξ, let NS_α^ξ be the set of non-ξ-s-stationary subsets of α.

Thus, if α has uncountable cofinality, NS_α^1 is the ideal of non-stationary subsets of α and $(NS_\alpha^1)^*$ is the club filter over α.

Notice that $\zeta \leq \xi$ implies $NS_\alpha^\zeta \subseteq NS_\alpha^\xi$ and $(NS_\alpha^\zeta)^* \subseteq (NS_\alpha^\xi)^*$.

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For each limit ordinal α and each ξ, let NS^{ξ}_{α} be the set of non-ξ-s-stationary subsets of α.

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Also note that $A \subseteq \alpha$ belongs to $(NS^{\xi}_{\alpha})^*$ if and only if for some $\zeta < \xi$ and some ζ-s-stationary sets $S, T \subseteq \alpha$, the set $D_{\zeta}(S) \cap D_{\zeta}(T) \cap \alpha$ is contained in A. In particular, if $S \subseteq \alpha$ is ζ-s-stationary, with $\zeta < \xi$, then $D_{\zeta}(S) \cap \alpha \in (NS^{\xi}_{\alpha})^*$.

Theorem

For every ξ, a limit ordinal α is ξ-s-stationary if and only if NS^{ξ}_{α} is a proper ideal, hence if and only if $(NS^{\xi}_{\alpha})^*$ is a proper filter.

Proof.

Assume α is ξ-s-stationary (hence $\alpha \notin NS^{\xi}_{\alpha}$) and let us show that NS^{ξ}_{α} is an ideal. For $\xi = 0$ this is clear. So, suppose $\xi > 0$ and $A, B \in NS^{\xi}_{\alpha}$. There exist $\zeta_A, \zeta_B < \xi$, and there exist sets S_A, $T_A \subseteq \alpha$ that are ζ_A-s-stationary in α, and sets S_B, $T_B \subseteq \alpha$ that are ζ_B-s-stationary in α, such that $D_{\zeta_A}(S_A) \cap D_{\zeta_A}(T_A) \cap A = D_{\zeta_B}(S_B) \cap D_{\zeta_b}(T_B) \cap B = \emptyset$. Hence,

$$(D_{\zeta_A}(S_A) \cap D_{\zeta_A}(T_A) \cap D_{\zeta_B}(S_B) \cap D_{\zeta_B}(T_B)) \cap (A \cup B) = \emptyset.$$

The set $X := D_{\zeta_A}(S_A) \cap D_{\zeta_A}(T_A) \cap D_{\zeta_B}(S_B) \cap D_{\zeta_B}(T_B)$ is $max\{\zeta_A, \zeta_B\}$-s-stationary in α. Now notice that

$$D_{max\{\zeta_A,\zeta_B\}}(X) \subseteq X$$

and so we have

$$D_{max\{\zeta_A,\zeta_B\}}(X) \cap \alpha \cap (A \cup B) = \emptyset$$

which witnesses that $A \cup B \in NS^{\xi}_{\alpha}$. □

Continued.

For the converse, assume NS^ξ_α is a proper ideal.
Take any A and B ζ-s-stationary subsets of α, for some $\zeta < \xi$. Then $D_\zeta(A) \cap \alpha$ and $D_\zeta(B) \cap \alpha$ are in $(NS^\xi_\alpha)^*$. Moreover, if $S, T \subseteq \alpha$ are any ζ'-s-stationary sets, for some $\zeta' < \xi$, then also $D_{\zeta'}(S) \cap \alpha$ and $D_{\zeta'}(T) \cap \alpha$ belong to $(NS^\xi_\alpha)^*$. Hence, since $(NS^\xi_\alpha)^*$ is a filter,

$$D_\zeta(A) \cap D_\zeta(B) \cap D_{\zeta'}(S) \cap D_{\zeta'}(T) \cap \alpha \in (NS^\xi_\alpha)^*$$

which implies, since $(NS^\xi_\alpha)^*$ is proper, that
$D_\zeta(A) \cap D_\zeta(B) \cap D_{\zeta'}(S) \cap D_{\zeta'}(T) \cap \alpha \neq \emptyset$. This shows that
$D_\zeta(A) \cap D_\zeta(B)$ is ξ-s-stationary in α. Since A and B were arbitrary, this implies α is ξ-s-stationary. □

Summary

The following are equivalent for every limit ordinal α and every $\xi > 0$:

1. α is a non-isolated point in the τ_ξ topology.
2. α is ξ-s-stationary.
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Indescribable cardinals

Recall that a formula of second-order logic is Σ^1_0 (or Π^1_0) if it does not have quantifiers of second order, but it may have any finite number of first-order quantifiers and free first-order and second-order variables.

Definition

For ξ any ordinal, we say that a formula is $\Sigma^1_{\xi+1}$ if it is of the form

$$\exists X_0, \ldots, X_k \varphi(X_0, \ldots, X_k)$$

where $\varphi(X_0, \ldots, X_k)$ is Π^1_ξ.
And a formula is $\Pi^1_{\xi+1}$ if it is of the form

$$\forall X_0, \ldots, X_k \varphi(X_0, \ldots, X_k)$$

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where $\varphi(X_0, \ldots, X_k)$ is Σ^1_ξ.

Definition

If ξ is a limit ordinal, then we say that a formula is Π^1_ξ if it is of the form

$$\bigwedge_{\zeta<\xi} \varphi_\zeta$$

where φ_ζ is Π^1_ζ, all $\zeta < \xi$, and it has only finitely-many free second-order variables. And we say that a formula is Σ^1_ξ if it is of the form

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where φ_ζ is Σ^1_ζ, all $\zeta < \xi$, and it has only finitely-many free second-order variables.

Definition

A cardinal κ is Π^1_ξ-*indescribable* if for all subsets $A \subseteq V_\kappa$ and every Π^1_ξ sentence φ, if

$$\langle V_\kappa, \in, A \rangle \models \varphi$$

then there is some $\lambda < \kappa$ such that

$$\langle V_\lambda, \in, A \cap V_\lambda \rangle \models \varphi.$$

Theorem

Every Π^1_ξ-indescribable cardinal is $(\xi+1)$-s-stationary. Hence, if ξ is a limit ordinal and a cardinal κ is Π^1_ζ-indescribable for all $\zeta<\xi$, then κ is ξ-s-stationary.

Proof.

Let κ be an infinite cardinal. Clearly, the fact that a set $A \subseteq \kappa$ is 0-s-stationary (i.e., unbounded) in κ can be expressed as a Π^1_0 sentence $\varphi_0(A)$ over $\langle V_\kappa, \in, A\rangle$. Inductively, for every $\xi > 0$, the fact that a set $A \subseteq \kappa$ is ξ-s-stationary in κ can be expressed by a Π^1_ξ sentence φ_ξ over $\langle V_\kappa, \in, A\rangle$. Namely,

$$\bigwedge_{\zeta<\xi} (A \text{ is } \zeta\text{-s-stationary})$$

in the case ξ is a limit ordinal, and by the sentence

$$\bigwedge_{\zeta<\xi-1} (A \text{ is } \zeta\text{-s-stationary}) \wedge$$

$$\forall S, T (S, T \text{ are } (\xi-1)\text{-s-stationary in } \kappa \rightarrow$$

$$\exists \beta \in A (S \text{ and } T \text{ are } (\xi-1)\text{-s-stationary in } \beta))$$

which is easily seen to be equivalent to a Π^1_ξ sentence, in the case ξ is a successor ordinal. □

Continued.

Now suppose κ is Π^1_ξ-indescribable, and suppose that A and B are ζ-s-stationary subsets of κ, for some $\zeta \leq \xi$. Thus,

$$\langle V_\kappa, \in, A, B \rangle \models \varphi_\zeta[A] \wedge \varphi_\zeta[B].$$

By the Π^1_ζ-indescribability of κ there exists $\beta < \kappa$ such that

$$\langle V_\beta, \in, A \cap \beta, B \cap \beta \rangle \models \varphi_\zeta[A \cap \beta] \wedge \varphi_\zeta[B \cap \beta]$$

which implies that A and B are ζ-s-stationary in β. Hence κ is $(\xi + 1)$-s-reflecting. □

Reflection and indescribability in L

Theorem (J.B.-M. Magidor-H. Sakai, 2013; J.B., 2015)

Assume $V = L$. For every $\xi > 0$, a regular cardinal is $(\xi + 1)$-stationary if and only if it is Π^1_ξ-indescribable, hence if and only if it is $(\xi + 1)$-s-stationary.[a][b]

[a] *Reflection and indescribability in the constructible universe*. Israel J. of Math. Vol. 208, Issue 1 (2015)

[b] *Derived topologies on ordinals and stationary reflection.* Preprint (2015)

The proof actually shows the following:

Theorem

Assume $V = L$*. Suppose* $\xi > 0$ *and* κ *is a regular* $(\xi+1)$*-stationary cardinal. Then for every* $A \subseteq \kappa$ *and every* Π^1_ξ *sentence* Ψ*, if* $\langle L_\kappa, \in, A\rangle \models \Psi$*, then there exists a* ξ*-stationary* $S \subseteq \kappa$ *such that* Ψ *reflects to every ordinal* λ *on which* S *is* ξ*-stationary.*

Theorem

CON($\exists\kappa<\lambda$ (κ *is* Π^1_ξ*-indescribable* $\wedge$ λ *is inaccessible*)) *implies* *CON*($\tau_{\xi+1}$ *is non-discrete* $\wedge$ $\tau_{\xi+2}$ *is discrete*).

Proof.

Let κ be Π^1_ξ-indescribable, and let $\lambda>\kappa$ be inaccessible. In L, κ is Π^1_ξ-indescribable and λ is inaccessible. So, in L, let κ_0 be the least Π^1_ξ-indescribable cardinal, and let λ_0 be the least inaccessible cardinal above κ_0. Then L_{λ_0} is a model of ZFC in which κ_0 is Π^1_ξ-indescribable and no regular cardinal greater than κ_0 is 2-stationary. The reason is that if α is a regular cardinal greater than κ_0, then $\alpha=\beta^+$, for some cardinal β. And since Jensen's principle $\Box_\beta$ holds, there exists a stationary subset of α that does not reflect. □

Theorem

CON($\exists\kappa < \lambda$ (κ *is* Π^1_ξ*-indescribable* $\wedge$ λ *is inaccessible*)) *implies* *CON*($\tau_{\xi+1}$ *is non-discrete* $\wedge$ $\tau_{\xi+2}$ *is discrete*).

Proof.

Let κ be Π^1_ξ-indescribable, and let $\lambda > \kappa$ be inaccessible. In L, κ is Π^1_ξ-indescribable and λ is inaccessible. So, in L, let κ_0 be the least Π^1_ξ-indescribable cardinal, and let λ_0 be the least inaccessible cardinal above κ_0. Then L_{λ_0} is a model of ZFC in which κ_0 is Π^1_ξ-indescribable and no regular cardinal greater than κ_0 is 2-stationary. The reason is that if α is a regular cardinal greater than κ_0, then $\alpha = \beta^+$, for some cardinal β. And since Jensen's principle $\square_\beta$ holds, there exists a stationary subset of α that does not reflect. $\square$

On the consistency strength of 2-stationarity

Let us write:

$$d_\xi(A) := \{\alpha : A \cap \alpha \text{ is } \xi\text{-stationary in } \alpha\}$$

Definition (A. H. Mekler-S. Shelah, 1989)

A regular uncountable cardinal κ is a reflection cardinal if there exists a proper, normal, and κ-complete ideal $\mathcal{I}$ on κ such that for every $X \subseteq \kappa$,

$$X \in \mathcal{I}^+ \quad \Rightarrow \quad d_1(X) \in \mathcal{I}^+.$$

Note: if κ is 2-stationary, then NS_κ is the smallest such ideal.
κ is weakly compact $\Rightarrow$ many reflection cardinals below κ.

On the consistency strength of 2-stationarity

Let us write:

$$d_\xi(A) := \{\alpha : A \cap \alpha \text{ is } \xi\text{-stationary in } \alpha\}$$

Definition (A. H. Mekler-S. Shelah, 1989)

A regular uncountable cardinal κ is a reflection cardinal if there exists a proper, normal, and κ-complete ideal $\mathcal{I}$ on κ such that for every $X \subseteq \kappa$,

$$X \in \mathcal{I}^+ \quad \Rightarrow \quad d_1(X) \in \mathcal{I}^+.$$

Note: if κ is 2-stationary, then NS_κ is the smallest such ideal.
κ is weakly compact $\Rightarrow$ many reflection cardinals below κ.

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If κ is a reflection cardinal in L, then in some generic extension of L that preserves cardinals, κ is 2-stationary. (In fact, the set Reg $\cap \kappa$ of regular cardinals below κ is 2-stationary).

Corollary

The following are equiconsistent:

1. *There exists a reflection cardinal.*
2. *There exists a 2-stationary cardinal.*
3. *There exists a regular cardinal κ such that every κ-free abelian group is κ^+-free.*

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Recall that a regular cardinal κ is greatly Mahlo if there exists a proper, normal, and κ-complete ideal $\mathcal{I}$ on κ such that $Reg \cap \kappa \in \mathcal{I}^*$, and for every $X \subseteq \kappa$,

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Theorem (A. H. Mekler-S. Shelah, 1989)

In L, if κ is at most the first greatly-Mahlo cardinal, then κ is not a reflection cardinal.

Thus, in L, the first reflection cardinal is strictly between the first greatly-Mahlo and the first weakly-compact.

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On the consistency strength of n-stationarity

We would like to prove analogous results for the n-stationay sets. So, let's define:

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For $n > 0$, a regular uncountable cardinal κ is an *n-reflection cardinal* if there exists a proper, normal, and κ-complete ideal $\mathcal{I}$ on κ such that for every $X \subseteq \kappa$,

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If κ is a 2-reflection cardinal in L, then in some generic extension of L that preserves cardinals, κ is 3-stationary.
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Similar arguments should yield a similar result for $n > 3$.

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A regular cardinal κ is *n-greatly Mahlo* if there exists a proper, normal, and κ-complete ideal $\mathcal{I}$ on κ such that $Reg \cap \kappa \in \mathcal{I}^*$, and for every $X \subseteq \kappa$,

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On the consistency strength of n-s-stationarity.

Magidor[1] shows that the following are equiconsistent:

1. There exists a 2-s-stationary cardinal (i.e., a cardinal that reflects simultaneously pairs of stationary sets).
2. There exists a weakly-compact cardinal.

Conjecture

The following should be equiconsistent for every $n > 0$:

1. *There exists an $(n+1)$-s-stationary cardinal.*
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[1] M. Magidor, On reflecting stationary sets. JSL 47 (1982).

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