Generalizing Schreier families to large index sets

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Outline

Main References

- S. A. Argyros and S. Todorcevic, *Ramsey methods in analysis*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2005.
- C. Brech, J. Lopez-Abad, and S. Todorcevic, *Homogeneous families on trees and subsymmetric basic sequences*, preprint.
- J. Lopez-Abad and S. Todorcevic, *Positional graphs and conditional structure of weakly null sequences*, Adv. Math. **242** (2013), 163–186.
- S. Todorcevic, *Walks on ordinals and their characteristics*, Progress in Mathematics, vol. 263, Birkhäuser Verlag, Basel, 2007.

Useful tools

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Fact 1

TFAE:

- *beer, wine, water, coffee, bread;*
- *pivo, víno, voda, káva, chléb/chleba.*

Basic notation and definitions

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- pre-compact if every sequence in $\mathcal{F}$ has a subsequence which forms a Δ-system;
- large if it contains arbitrarily large (in cardinality) finite subsets within any infinite subset X of I.

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Example 3 (Schreier family)

The family $\mathcal{S} = \{\emptyset\} \cup \{s \in [\omega]^{<\omega} : |s| \leq \min s + 1\}$ is hereditary, compact and large.

Indiscernibles

In model theory, a set of indiscernibles for a given structure $\mathcal{M}$ is a subset X with a total order $<$ such that, for every positive integer n, every two increasing n-tuples $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$ of elements of X have the same properties in $\mathcal{M}$.

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If $\mathcal{F}$ is a compact large family on I, then the relational structure $\mathcal{M}_{\mathcal{F}} := (I, (\mathcal{F} \cap [I]^n)_n)$ has no infinite sets of indiscernibles.

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Indiscernibles in Banach spaces: subsymmetric sequences

A sequence $(x_n)_n$ in a Banach space X is subsymmetric if there is $C \geq 1$ such that for all $(\lambda_i)_{i=1}^{l}$ and all increasing sequences $(k_i)_{i=1}^{l}$ and $(n_i)_{i=1}^{l}$ we have that

$$\|\sum_{i=1}^{l} \lambda_i x_{k_i}\| \leq C\|\sum_{i=1}^{l} \lambda_i x_{n_i}\|.$$

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Example 4

The unit bases of c_0 and ℓ_p, $1 \leq p < \infty$ are (sub)symmetric.

Indiscernibles in Banach spaces: subsymmetric sequences

Example 5 (Schreier space)

Given: $x = (x_n)_n \in c_{00}(\omega)$, let $\|x\|_{\mathcal{S}} = \sup\{\sum_{n \in s} |x_n| : s \in \mathcal{S}\}$.
$\|\cdot\|_{\mathcal{S}}$ is a norm and the completion of $(c_{00}(\omega), \|\cdot\|_{\mathcal{S}})$ is a Banach space such that $(e_n)_n$ is an unconditional basis with no subsymmetric basic subsequences.

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Lemma 6 (Pták, 1963)

If $\mathcal{F}$ is a compact family on ω, then for every $\varepsilon > 0$, there is a finite $F \subseteq \omega$ and positive $(a_\alpha)_{\alpha \in F}$ such that $\sum_{\alpha \in F} a_\alpha = 1$ and $\sum_{\alpha \in s} a_\alpha < \varepsilon$ if $s \in \mathcal{F} \cap \wp(F)$.

Lopez-Abad and Todorcevic result

Theorem 7 (Lopez-Abad, Todorcevic, 2013)

Let κ be an infinite cardinal. TFAE:

(a) *κ is not ω-Erdös, i.e., if $\kappa \not\to (\omega)_2^{<\omega}$;*

(b) *there is a hereditary, compact and large family $\mathcal{F}$ on κ;*

(c) *there is a nontrivial normalized weakly-null basis $(x_\alpha)_{\alpha<\kappa}$ in a Banach space with no subsymmetric basic subsequence.*

(a) implies (b)

Fact 8

If $\kappa \not\to (\omega)_2^{<\omega}$ *and* $c : [\kappa]^{<\omega} \to 2$, *then*

$$\mathcal{F}_c = \{s \subseteq \omega : s \text{ is monochromatic}\}$$

is a hereditary, compact and large family on κ.

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is a hereditary, compact and large family on κ.

Proof.

It is clearly hereditary and it is easy to check that it is compact. Largeness is a consequence of the finite Ramsey theorem. The fact that $\kappa \not\to (\omega)^{<\omega}_2$ is needed only to guarantee that $\mathcal{F}_c$ consists of finite subsets of κ. □

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Fact 9

If $\mathcal{F}$ *is a hereditary, compact and large family on* κ *and* $x = (x_\alpha)_\alpha \in c_{00}(\kappa)$, *let*

$$\|x\|_{\mathcal{F}} = \sup\{\sum_{\alpha \in s} |x_\alpha| : s \in \mathcal{F}\}.$$

$\|\cdot\|_{\mathcal{F}}$ *is a norm and the completion of* $(c_{00}(\kappa), \|\cdot\|_{\mathcal{F}})$ *is a Banach space such that* $(e_\alpha)_\alpha$ *is an unconditional basis with no subsymmetric basic subsequences.*

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Proof.

Analogous to the Schreier space. □

(c) implies (a)

Exercise 2

$\kappa \to (\omega)^{<\omega}_2$ iff $\kappa \to (\omega)^{<\omega}_{2^\omega}$.

Hint: Given $c : [\kappa]^{<\omega} \to 2^\omega$ and $\theta : \omega^2 \to \omega$ bijection such that $\theta(i,j) \geq i$, let $d : [\kappa]^{<\omega} \to 2$ be such that $d(s)$ is the j-th coordinate of the c-color of the subset of s consisting of its first i-many elements, where $\theta(i,j) = |s|$ and show that a d-monochromatic set is also c-monochromatic.

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Fact 10 (Ketonen, 1974)

Given $(x_\alpha)_{\alpha<\kappa}$, for each $s = \{\alpha_1 < \cdots < \alpha_n\} \in [\kappa]^{<\omega}$ with $|s| = n$, define f_s on $\mathbb{R}^n$ by $f_s(a_1, \ldots, a_n) = \|a_1 x_{\alpha_1} + \cdots + a_n x_{\alpha_n}\|$ and define $c : [\kappa]^{<\omega} \to \bigcup_{n\in\omega}\{n\} \times \mathbb{R}^{n+1}$ by $c(s) = (|s|, f_s)$. If A is an infinite monochromatic subset of κ, then $(x_\alpha)_{\alpha\in A}$ is symmetric.

Tsirelson space

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Example 11 (Tsirelson space)

Given $x = (x_n)_n \in c_{00}(\omega)$, let $\|x\|_T$ on $c_{00}(\omega)$ be such that

$$\|x\|_T = \sup\{\|x\|_\infty, \frac{1}{2}\sum_{i=1}^{n} \|\langle x_i, \chi_{s_i}\rangle\|_T : s_i < s_{i+1},\ \{\min s_i\}_{1\leq i\leq n} \in \mathcal{S}\}.$$

$\|\cdot\|_T$ is a norm and the completion of $(c_{00}(\omega), \|\cdot\|_T)$ is a (separable) reflexive Banach space with no subsymmetric basic sequences.

Nonseparable Tsirelson-like spaces

However, the natural nonseparable version of the Tsirelson space, replacing the Schreier family by a hereditary compact and large family on an uncountable cardinal κ, yields a space with copies of ℓ_1, hence with subsymmetric basic sequences (Lopez-Abad, Todorcevic, 2013).

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If $\mathcal{F}$ is a large and spreading family on an uncountable index set, then $\mathcal{F}$ is not compact.

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If $\mathcal{F}$ is a large and spreading family on an uncountable index set, then $\mathcal{F}$ is not compact.

To overcome this obstacle, we switch from a single large family to sequences of families obtained by making some kind of products by families on ω, such as the Schreier family.

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Given a family $\mathcal{F}$ on a cardinal κ and a family $\mathcal{H}$ on ω, we say that a family $\mathcal{G}$ on κ is a multiplication of $\mathcal{F}$ by $\mathcal{H}$ if every infinite sequence $(s_n)_n$ in $\mathcal{F}$ has an infinite subsequence $(t_n)_n$ such that, for every $x \in \mathcal{H}$, $\bigcup_{n\in x} t_n \in \mathcal{G}$.

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We say that a sequence of families $(\mathcal{F}_n)_n$ on κ is a CL-sequence (consecutively large sequence) of families on κ if each family is hereditary and compact and $\mathcal{F}_{n+1}$ is a multiplication of $\mathcal{F}_n$ by $\mathcal{S}$.

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Theorem 13 (B., Lopez-Abad, Todorcevic)

For every infinite cardinal κ smaller than the first Mahlo cardinal, there is a CL-sequence of families on κ.

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Recall that a cardinal κ is Mahlo if it is strongly inaccessible and $\{\lambda < \kappa : \lambda \text{ is strongly inaccessible}\}$ is stationary.

Nonseparable Tsirelson-like spaces

Theorem 14 (B., Lopez-Abad, Todorcevic & Argyros, Motakis)

If $(\mathcal{F}_n)_n$ is a CL-sequence, then there is a Banach space X of density κ with an unconditional (long) basis and with no subsymmetric sequences.

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Sketch.

Given $x \in c_{00}(\kappa)$, let

$$\|x\| = \sup\{\|x\|_\infty, \|\sum_{n=0}^{\infty} \frac{\|x\|_{\mathcal{F}_n}}{2^{n+1}}\|_T\}.$$

This is a norm such that the closure with respect to it is a Banach space of density κ with an unconditional basis and with no subsymmetric sequences. □

A CL-sequence on ω

Example 15

Given hereditary and compact families $\mathcal{F}$ and $\mathcal{F}'$ on ω, let

$$\mathcal{F} \oplus \mathcal{F}' = \{s \cup t : s < t,\ s \in \mathcal{F}',\ t \in \mathcal{F}\},$$

$$\mathcal{F} \otimes \mathcal{F}' = \{\bigcup_{k<n} s_k : n \in \omega,\ s_k < s_{k+1},\ s_k \in \mathcal{F},\ \{\min s_k : k < n\} \in \mathcal{F}'\},$$

and notice that $\mathcal{G} = (\mathcal{F} \otimes \mathcal{S}) \oplus \mathcal{F}$ is a compact and hereditary family on ω and a multiplication of $\mathcal{F}$ by $\mathcal{S}$.

A CL-sequence on ω

Example 15

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and notice that $\mathcal{G} = (\mathcal{F} \otimes \mathcal{S}) \oplus \mathcal{F}$ is a compact and hereditary family on ω and a multiplication of $\mathcal{F}$ by $\mathcal{S}$.
Define inductively:

- $\mathcal{F}_0 = \mathcal{S}$;
- $\mathcal{F}_{n+1} = (\mathcal{S}_n \otimes \mathcal{S}) \oplus \mathcal{S}_n$.

$(\mathcal{F}_n)_n$ is a CL-sequence of families on ω.