Steinhaus properies of σ-ideals in Polish groups

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σ-Ideals on groups

A family $\mathcal{I}$ of subsets of a set X is called a *σ-ideal* if $\mathcal{I}$ is closed under countable unions and taking subsets.

Example

Standard examples of σ-ideals are:

- $\mathcal{M}$, the ideal of meager sets;
- $\mathcal{N}$, the ideal of (Lebesgues) null sets;
- $\mathcal{E}$, the σ-ideal generated by closed null sets in $\mathbb{R}$

(more generally, in a Polish locally compact group G).

These three σ-ideals relate as follows:

$$\begin{array}{ccccc} \mathcal{M} & \longleftarrow & \mathcal{M} \cap \mathcal{N} & \longrightarrow & \mathcal{N} \\ & & \uparrow & & \\ & & \mathcal{E} & & \end{array}$$

0-1-Theorems of Banach, Steinhaus, Laczkovich

Theorem (S.Banach, 1931)

Any Borel subgroup H of a Polish group G is either open or belongs to the σ-ideal $\mathcal{M}$ of meager sets in G.

Theorem (H.Steinhaus, 1920)

Any Borel subgroup H of a Polish locally compact group G is either open or belongs to the σ-ideal $\mathcal{N}$ of Haar null sets in G.

Theorem (Laczkovich, 1998)

Any Borel subgroup H of a Polish locally compact group G is either open or belongs to the σ-ideal $\mathcal{E}$ generated by closed Haar null sets in G.

General Problem

Problem

Detect σ-ideals $\mathcal{I}$ on a Polish group containing all non-open Borel subgroups of the group.

We shall show that a σ-ideal $\mathcal{I}$ has this property if $\mathcal{I}$ is $\pm n$-Steinhaus for some $n \in \mathbb{N}$.
Before defining $\pm n$-Steinhaus ideals, let us discuss the reason why Banach and Steinhaus Theorems are true.

Theorems of Steinhaus and Piccard-Pettis

Theorems of Steinhaus and Banach are immediate corollaries of the following two classical results:

Theorem (Steinhaus, 1920)

For any measurable sets $A, B \notin \mathcal{N}$ in a locally compact group G the product $AB = \{ab : a \in A,\ b \in B\}$ has non-empty interior and the difference AA^{-1} is a neighborhood of the unit in G.

Theorem (Piccard, 1939; Pettis, 1951)

For any BP-sets $A, B \notin \mathcal{M}$ in a Polish group G the product $AB = \{ab : a \in A,\ b \in B\}$ has non-empty interior and the difference AA^{-1} is a neighborhood of the unit in G.

A set A of a topological space X has the *Baire property* (or is a *BP-set*) if there exists an open set $U \subset X$ such that the symmetric difference $A \Delta U$ is meager in X.

n-Steinhaus ideals

Definition

An ideal $\mathcal{I}$ of subsets of a topological group G is *n-Steinhaus* for some $n \in \mathbb{N}$ if for any subsets $A_1, \dots, A_n \notin \mathcal{I}$ the product $A_1 \cdots A_n$ is not nowhere dense (= the closure $\overline{A_1 \cdots A_n}$ has non-empty interior) in G.

An equivalent reformulation:

Definition

An ideal $\mathcal{I}$ of subsets of a topological group G is *n-Steinhaus* for some $n \in \mathbb{N}$ if for any closed subsets $A_1, \dots, A_n \notin \mathcal{I}$ the product $A_1 \cdots A_n$ is not nowhere dense (= the closure $\overline{A_1 \cdots A_n}$ has non-empty interior) in G.

The reason: $\overline{\bar{A}_1 \cdots \bar{A}_n} = \overline{A_1 \cdots A_n}$.

$$n\text{-Steinhaus} \Rightarrow (n+1)\text{-Steinhaus}$$

For a subset A of a group G put $A^{\pm} = A \cup A^{-1}$.

Definition

An ideal $\mathcal{I}$ of subsets of a topological group G is *$\pm n$-Steinhaus* if for any (closed) subsets $A_1, \dots, A_n \notin \mathcal{I}$ the set $A_1^{\pm} \cdots A_n^{\pm}$ is not nowhere dense in G.

$$\begin{array}{ccc} n\text{-Steinhaus} & \Longrightarrow & (n+1)\text{-Steinhaus} \\ \Downarrow & & \Downarrow \\ \pm n\text{-Steinhaus} & \Longrightarrow & \pm(n+1)\text{-Steinhaus}. \end{array}$$

The Steinhaus properties of the classical ideals

For any Polish locally compact group G the σ-ideal

- $\mathcal{M}$ is 1-Steinhaus;
- $\mathcal{N}$ is 2-Steinhaus;
- $\mathcal{M} \cap \mathcal{N}$ is 2-Steinhaus;
- $\mathcal{E}$ is 2-Steinhaus.

$\pm n$-Steinhaus σ-ideals contain meager analytic subgroups

A space A is *analytic* if A is a continuous image of a Polish space.

Theorem (B-K-R, 2015)

Any non-open analytic subgroup H of a Polish group G belongs to any $\pm n$-Steinhaus σ-ideal on G.

Corollary (Laczkovich, 1998)

Any non-open analytic subgroup H of a Polish locally compact group G belongs to the σ-ideal $\mathcal{E}$.

We shall derive the theorem from:

Theorem (BKR, 2015)

Any meager quasi-analytic subgroup H of a Polish group G belongs to any $\pm n$-Steinhaus σ-ideal $\mathcal{I}$ on G.

A topological space is *quasi-analytic* if it is a continuous image of a metrizable separable hereditarily Baire space.

A characterization of $\pm n$-Steinhaus property

Theorem (BKR, 2015)

Any meager quasi-analytic sub(semi)group H of a Polish group G belongs to any $\pm n$-Steinhaus (n-Steinhaus) σ-ideal $\mathcal{I}$ on G.

This theorem will be derived from two characterizations:

Theorem (BKR, 2015)

A σ-ideal $\mathcal{I}$ on a Polish group G is $\pm n$-Steinhaus iff for any quasi-analytic sets $A_1, \dots, A_n \notin \mathcal{I}$ the product $A_1^{\pm} \dots A_n^{\pm}$ is non-meager.

Theorem (BKR,2015)

A σ-ideal $\mathcal{I}$ on a Polish group G is n-Steinhaus iff for any quasi-analytic sets $A_1, \dots, A_n \notin \mathcal{I}$ the product $A_1 \cdots A_n$ is non-meager.

Proof

Given an n-Steinhaus σ-ideal $\mathcal{I}$ on a top.group G and quasi-analytic sets $A_1, \dots, A_n \notin \mathcal{I}$ of G, we need to prove that $A_1 \cdots A_n$ is not meager in G. To derive a contradiction, assume that $A_1 \cdots A_n$ is meager and hence is contained in a union $\bigcup_{m \in \omega} F_m$ of closed nowhere dense sets $F_m \subset G$. Write each quasi-analytic space A_i as a continuous image of a hereditarily Baire metrizable separable space B_i under a continuous map $f_i : B_i \to A_i$. Let $\mathcal{U}_i$ be the family of all open sets $U \subset B_i$ with $f_i(U) \in \mathcal{I}$. Then $U_i = \bigcup \mathcal{U}_i \in \mathcal{U}_i$ and $B_i' = B_i \setminus U_i$ is not empty, Baire, and for every non-empty open set $V \subset B_i'$ we get $f_i(V) \notin \mathcal{I}$. Consider the map $\pi : B_1' \times \cdots \times B_n' \to G$, $\pi : (x_1, \dots, x_n) \mapsto f_1(x_1) \cdots f_n(x_n)$, and observe that $B = B_1' \times \cdots \times B_n'$ is a Baire space. By Baire Theorem, for some $m \in \omega$ the closed set $\pi^{-1}(F_m)$ has non-empty interior in B and hence contains some open set $V_1 \times \cdots \times V_n$. The choice of B_i' guarantees that the sets $A_i' = f_i(V_i) \notin \mathcal{I}$ and hence $A_1' \cdots A_n'$ is not nowhere dense in G. On the other hand, $A_1' \cdots A_n' = \pi(V_1 \times \cdots \times V_n) \subset F_m$ is nowhere dense. This contradiction completes the proof.

Quasi-analytic spaces

We recall that a space X is *quasi-analytic* if X is a continuous image of a hereditarily Baire metrizable separable space.
It is clear that each analytic space is quasi-analytic.
A Bernstein set on the real line is an example of a quasi-analytic (in fact, a hereditarily Baire) space which is not analytic.
It is not so easy to find a metrizable separable space, which is not quasi-analytic.
The simplest example is any uncountable universally meager set.
A subset A of the real line $\mathbb{R}$ is *universaly meager* if for any nowhere locally constant map $f : P \to \mathbb{R}$ from a Polish space the preimage $f^{-1}(A)$ is meager.
It is known that universally meager sets exist in ZFC but in some models universally meager sets have cardinality $\leq \aleph_1 < \mathfrak{c}$.

Theorem (B., Zdomskyy, 2016)

Each uncountable quasi-analytic space has cardinality $\mathfrak{c}$.

Steinhaus properties of σ-ideals $\mathcal{M}_{|K}$

For a Baire subspace K of a topological group G by $\mathcal{M}_{|K}$ denote the σ-ideal generated by closed subsets $A \subset G$ such that for every $x \in G$ the intersection $K \cap xA$ is meager in K.
For example, the ideal $\mathcal{M}_{|[0,1]}$ coinicides with the ideal $\mathcal{M}$ on $\mathbb{R}$.

Proposition

Let K be a Baire subspace of an abelian topological group. The ideal $\mathcal{M}_{|K}$ is n-Steinhaus (resp. $\pm$n-Steinhaus) for some $n \in \mathbb{N}$ iff for any non-empty open sets $U_1, \dots, U_n \subset K$ the product $U_1 \cdots U_n$ (resp. $U_1^{\pm} \cdots U_n^{\pm}$) is not nowhere dense in G.

Corollary

Let $G = \prod_{i\in\omega} G_i$ be the Tychonoff product of discrete abelian groups and $K = \prod_{i\in\omega} K_i \subset G$. The σ-ideal $\mathcal{M}_{|K}$ is n-Steinhaus (resp. $\pm$n-Steinhaus) if and only if $(K_i)^n = G_i$ (resp. $(K_i^{\pm})^n = G_i$) for all but finitely many $i \in \omega$.

Distinguishing examples of Steinhaus ideals

Example

For every $n \in \mathbb{N}$ there exists a compact topological group G and a σ-ideal $\mathcal{I}$ which is $(n+1)$-Steinhaus but not $\pm n$-Steinhaus.

Proof.

Consider the cyclic group $\mathbb{Z}_{2n+3} = \{-n-1, \dots, 0, \dots, n+1\}$ and the 3-element set $\{-1, 0, 1\}$ in $\mathbb{Z}_{2n+3}$. For the compact set $K = \{-1, 0, 1\}^\omega$ the ideal $\mathcal{M}_{|K}$ in the group $G = \mathbb{Z}_{2n+3}^\omega$ is $(n+1)$-Steinhaus but not $\pm n$-Steinhaus. □

Another strange example

Example

There exists a compact topological groups G and a σ-ideal $\mathcal{I} = \mathcal{I}^{-1}$ which is ± 2-Steinhaus but is n-Steinhaus for no $n \in \mathbb{N}$.

Proof.

By a result of Haight (1973) for every $n \in \mathbb{N}$ there exists a subset K_n in a finite cyclic group G_n such that $K_n K_n^{-1} = G_n$ but $(K_n)^n \neq G_n$. Then for the compact set $K = \prod_{n\in\omega} K_n$ the σ-ideal $\mathcal{I} = \mathcal{M}_{|K}$ is ± 2-Steinhaus but is not n-Steinhaus for all $n \in \mathbb{N}$. Moreover, for the symmetric set $K^{\pm} = K \cup K^{-1}$ the ideal $\mathcal{I} = \mathcal{M}_{|K^{\pm}}$ is equal to $\mathcal{I}^{-1}$, is ± 2-Steinhaus but is n-Steinhaus for no $n \in \mathbb{N}$. □

Theorem

Any meager quasi-analytic subsemigroup H of a Polish locally compact group G belongs to the σ-ideal $\mathcal{E}$ generated by closed Haar null sets.

Problem

Can this theorem be generalized to non-locally compact groups?

Problem

Can the σ-ideals $\mathcal{N}, \mathcal{E}$ be defined in non-locally compact groups?

Yes! In many different ways!

Definition (Chrystensen, 1973)

A Borel subset B of a Polish group G is *Haar null* if there exists a Borel probability measure μ on G such that $\mu(xB) = 0$ for all $x \in G$.

Theorem (Chrystensen, 1973)

The family $\mathcal{N}_{Ch}$ of all subsets of Borel Haar null sets in an abelian Polish group G is a σ-ideal containing all non-open Borel subgroups of G.

This follows from:

Theorem (Chystensen, 1973)

If a Borel subset B of an abelian Polish group G is not Haar null, then BB^{-1} is a neighborhood of the unit in G.

Generically Haar null sets of Dodos

Definition (Dodos, 2004)

A Borel subset B of a Polish group G is *generically Haar null* if the set $\{\mu \in P(G) : \forall x \in G\ \mu(xB) = 0\}$ is comeager in the space $P(G)$ of probability measures on G.

Theorem (Dodos, 2004)

The family $\mathcal{N}_{Ds}$ of subsets of Borel generically Haar null subsets of a Polish group G is a σ-ideal containing all non-open Borel subgrops of G.

This follows from:

Theorem (Dodos, 2004)

If an analytic subset A of a Polish group G is not generically Haar null, then AA^{-1} is not meager and $AA^{-1}AA^{-1}$ is a neighborhood of the unit in G. Consequently, each non-open analytic subgroup H of G is generically Haar null.

Haar-Meager sets of Darji

Definition (Darji, 2013)

A Borel subset B of a Polish group G is *Haar meager* if there exists a map $f : 2^\omega \to G$ such that for every $x \in G$ the set $f^{-1}(xA)$ is meager in 2^ω.

Theorem (Darji, Jablonska)

The family $\mathcal{M}_{Dj}$ of subsets of Borel Haar-meager subsets of an abelian Polish group G is a σ-ideal containing all non-open Borel subgroups of G.

The latter statement follows from:

Theorem (Jablonska, 2015)

If a Borel subset A of an abelian Polish group G is not Haar-meager, then AA^{-1} is a neighborhood of the unit in G. Consequently, each non-open analytic subgroup H of G is Haar-meager.

Generically Haar-meager sets of B.-K.R.

Definition (B.-K.-R., 2015)

A subset B of a Polish group G is *generically Haar meager* if the set $\{K \in \mathcal{K}(G) : \forall x \in G \; xA \cap K \text{ is meager in } K\}$ is comeager in the hyperspace $\mathcal{K}(X)$ of all non-empty compact sets in X.

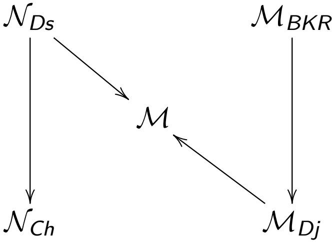
Theorem (B.-K.-R., 2015)

The family $\mathcal{M}_{BKR}$ of all subsets of Borel generically Haar meager subsets of a Polish group G is a σ-ideal containing all non-open Borel subgroups of G.

This follows from:

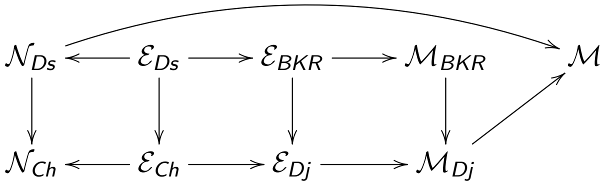
Theorem

If an analytic subset A of an abelian Polish group G is not generically Haar-meager, then AA^{-1} is non-meager and $AA^{-1}AA^{-1}$ is a neighborhood of the unit in G.

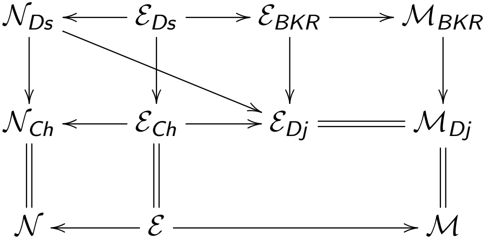

$\mathcal{N}_{Ds}$
$\mathcal{M}_{BKR}$
$\mathcal{M}$
$\mathcal{N}_{Ch}$
$\mathcal{M}_{Dj}$

σ-Ideals $\mathcal{E}_{Ch}$, $\mathcal{E}_{Ds}$, $\mathcal{E}_{Dj}$, $\mathcal{E}_{BKR}$

For a Polish group G let $\mathcal{E}_{Ch}$, $\mathcal{E}_{Ds}$, $\mathcal{E}_{Dj}$, $\mathcal{E}_{BKR}$ be the σ-ideals generated by closed subsets in the ideals $\mathcal{N}_{Ch}$, $\mathcal{N}_{Ds}$, $\mathcal{M}_{Dj}$, $\mathcal{M}_{BKR}$, respectively.



These ideals in Polish locally compact groups



0-1-Laws

Theorem

Every non-open Borel subgroup of a Polish group belongs to the σ-ideals $\mathcal{N}_{Ds} \subset \mathcal{N}_{Ch}$ and $\mathcal{M}_{BKR} \subset \mathcal{M}_{Dj} \subset \mathcal{M}$.

Theorem of Steinhaus and Piccard-Pettis imply:

Theorem

Every Borel subsemigroup with empty interior in a Polish locally compact group belongs to the σ-ideals $\mathcal{E} \subset \mathcal{M} \cap \mathcal{N}$.

Example

The nowhere dense subsemigroup $\mathbb{N}^\omega$ in the Polish group $\mathbb{Z}^\omega$ does not belong to the σ-ideals $\mathcal{N}_{Ch}$ and $\mathcal{M}_{Dj}$ (because $\mathbb{N}^\omega$ is thick in the sense that for each compact subset $K \subset \mathbb{Z}^\omega$ there is a shift $x + K \subset \mathbb{N}^\omega$).

Theorem (Laczkovich, 1998)

Any Borel subgroup H of a Polish locally compact group G belongs to the ideal $\mathcal{E}$.

Example (B.-K.-R., 2015)

The Polish group $\mathbb{Z}^\omega$ contains a Borel (in fact, σ-Polish) subgroup H which belongs to the σ-ideals $\mathcal{N}_{Ds} \cap \mathcal{M}_{BKR}$ but not to the σ-ideal $\mathcal{E}_{Dj}$ generated by closed Haar-meager sets (of Darji).

An Open Problem

Problem

Is there a ZFC-example of a subgroup $H \in \mathcal{M} \cap \mathcal{N} \setminus \mathcal{E}$ of the compact group $\mathbb{Z}_2^\omega$?

Theorem (Talagrand, 1982)

There is a ZFC-example of a subgroup $H \in \mathcal{N} \setminus \mathcal{M}$ of $\mathbb{Z}_2$.

Theorem (Burke, 198?)

There is no ZFC-example of a subgroup $H \in \mathcal{M} \setminus \mathcal{N}$ of $\mathbb{Z}_2^\omega$.

References

T.Banakh, L.Karchevska, A.Ravsky, *The closed Steinhaus properties of -ideals on topological groups*, preprint (http://arxiv.org/abs/1509.09073).

Thank You!