Filters and sets of Vitali type

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Introduction

Definition

Let $\mathcal{F}$ be a nonprincipal filter on ω. For points $x = (x_n)_{n\in\omega}, y = (y_n)_{n\in\omega} \in \{0,1\}^\omega$ we define ralation:

$$x \approx_{\mathcal{F}} y \text{ iff } \{n \in \omega : x_n = y_n\} \in \mathcal{F}$$

Clearly $\approx_{\mathcal{F}}$ is equevalence relation.

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Clearly $\approx_{\mathcal{F}}$ is equevalence relation.

1. $[x]_{\mathcal{F}}$ =abstract class of point x
2. $\{0,1\}/\mathcal{F}$ =set of all abstract classes of relation $\approx_{\mathcal{F}}$
3. We consider only nonprincipal filters.
4. $\mathbf{0} = (0,0,0,...)$ and $\mathbf{1} = (1,1,1,...)$
5. $-x = x + \mathbf{1}$

Basic properties

For any filters $\mathcal{F}_0, \mathcal{F}_1$ and $\mathcal{F}$ on ω we have

1. If $\mathcal{F}_0 \subset \mathcal{F}_1$ then $[x]_{\mathcal{F}_0} \subset [x]_{\mathcal{F}_1}$ for each $x \in \{0,1\}^\omega$
2. If $\mathcal{F}_0 \subset \mathcal{F}_1$ then each selector of $\{0,1\}^\omega/\mathcal{F}_1$ can be extended to selector of $\{0,1\}^\omega/\mathcal{F}_0$
3. If $F_0 \subset \mathcal{F}_1$ then each element of $\{0,1\}^\omega/\mathcal{F}_1$ is disjoit union of some elements of $\{0,1\}^\omega/\mathcal{F}_0$
4. Abstract class $[\mathbf{0}]_{\mathcal{F}}$ is a dense subgroup of $\{0,1\}^\omega$
5. For each $x \in \{0,1\}^\omega$ we have $[x]_{\mathcal{F}} = [\mathbf{0}]_{\mathcal{F}} + x$ and $[-x]_{\mathcal{F}} = [\mathbf{1}]_{\mathcal{F}} + x$

Measurability

Selector S of $\{0,1\}^\omega/\mathcal{F}$ is $\subset$-maximal set such that for two distinct points $x, y \in S$ we have $x + y \notin [\mathbf{0}]_{\mathcal{F}}$.

Fact

If $\mathcal{F}$ is filter on ω, $\mathbf{I}$ is an ideal with Steinhaus property and S is selector of $\{0,1\}^\omega/\mathcal{F}$ then

$$S \text{ is } \mathbf{I}\text{-measureable} \Rightarrow S \in \mathbf{I}$$

Fact

If $\mathcal{F}$ is filter on ω, $\mathbf{I}$ is an ideal with Steinhaus property and A is element of $\{0,1\}^\omega/\mathcal{F}$ then

$$A \text{ is } \mathbf{I}\text{-measureable} \Rightarrow A \in \mathbf{I}$$

Definition

We say that subset X of classical Vitali V set is consistent if there exist ultrafilter $\mathcal{U}$ such that $X \subset [\mathbf{0}]_{\mathcal{U}}$.
The smallest filter $\mathcal{F}$ with $X \subset [\mathbf{0}]_{\mathcal{F}}$ we denote by $\mathcal{F}_X$.
For filter $\mathcal{F}$ we define $V_{\mathcal{F}} = [\mathbf{0}]_{\mathcal{F}} \cap V$.

lemma

For any filter $\mathcal{F}$ we have $\mathcal{F} = \mathcal{F}_{V_{\mathcal{F}}}$

Definition

For consistent $X \subset V$ we define set of forbidden points for X as follows

$$Forb(X) = \{y \in \{0,1\}^\omega : X \cup \{y\} \text{ is not consistent}\}$$

lemma

$-Forb(X) = \{y \in \{0,1\}^\omega : -y \in Forb(X)\} \subset [0]_{\mathcal{F}(X)}$

Theorem

For a filter $\mathcal{F}$ if we have $x \notin [\mathbf{0}]_{\mathcal{F}} \cup [\mathbf{1}]_{\mathcal{F}}$ then both set

$$\{x\} \cup V_{\mathcal{F}} \text{ and } \{-x\} \cup V_{\mathcal{F}}$$

are consistent.

Proof. If for example $\{x\} \cup V_{\mathcal{F}}$ is not consistent then $x \in Forb(V_{\mathcal{F}})$ and then from second lemma we have $-x \in -Forb(V_{\mathcal{F}}) \subset [0]_{\mathcal{F}(V_{\mathcal{F}})}$ and from first lemma we see that $x \in [\mathbf{1}]_{\mathcal{F}(V_{\mathcal{F}})} = [\mathbf{1}]_{\mathcal{F}}$.

Theorem

Every consistent set $X \subset V$ can be extended to maximal with respect to inclusion consistent.

Theorem

For any $n \in \omega$ there exist a filter $\mathcal{F}$ such that $|\{0,1\}^\omega/\mathcal{F}| = 2^n$.
If $n \in \omega$ is not power of two thent there is no filter $\mathcal{F}$ such that $|\{0,1\}^\omega/\mathcal{F}| = n$.

Proof. For first part let $\mathcal{U}_1, ..., \mathcal{U}_n$ be distinct ultrafilters then $\mathcal{U} = \bigcap_{i=1}^n \mathcal{U}_i$ is filter with 2^n abstract classes.

$$\{0,1\}^\omega/\mathcal{U} = \{\bigcap_{i=1}^n [a_i]_{\mathcal{U}_i} : (a_i)_{i=1}^n \in \{0,1\}^n\}$$

For second part let $\mathcal{F}$ be a filter with $|\{0,1\}^\omega/\mathcal{F}| = n$ and let $\{x_k, -x_k : k = 0, 1, ..., n/2\}$ be a selector of $\{0,1\}^\omega/\mathcal{F}$ with $x_0 = \mathbf{0}$. For any $k = 1, ..., n/2$ both sets

$$\{x_k\} \cup V_{\mathcal{F}} \text{ and } \{-x_k\} \cup V_{\mathcal{F}}$$

are consistent.

Thus from previous there exist two ultrafilters $\mathcal{U}_0^k \neq \mathcal{U}_1^k$ which both extends $\mathcal{F}$ and also

$$\{x_k\} \cup V_{\mathcal{F}} \subset [\mathbf{0}]_{\mathcal{U}_0^k} \text{ and } \{-x_k\} \cup V_{\mathcal{F}} \subset [\mathbf{0}]_{\mathcal{U}_1^k} \text{ (star)}$$

Now as we already know from lemma above that

$$\mathcal{U} = \bigcap_{k=1}^{n/2}(\mathcal{U}_0^k \cap \mathcal{U}_1^k)$$

is a filter which extends $\mathcal{F}$ and has 2^m abstract classes for some $m \in \omega$ (we dont know if all $\{\mathcal{U}_0^k, \mathcal{U}_1^k : k = 1, ..., n/2\}$ are distinct).
Moreover we claim that $\mathcal{U} = \mathcal{F}$. We know that $\mathcal{F} \subset \mathcal{U}$ so let assume that inclusion is proper. Then exist $k \in \{1, ..., n/2\}$ with (wlog) $x_k \in [\mathbf{0}]_{\mathcal{U}}$ but then

$$x_k \in [\mathbf{0}]_{\mathcal{U}} = \bigcap_{i=1}^{n/2}[\mathbf{0}]_{\mathcal{U}_0^i} \cap [\mathbf{0}]_{\mathcal{U}_1^i}$$

so in particular $x_k \in [\mathbf{0}]_{\mathcal{U}_0^k} \cap [\mathbf{0}]_{\mathcal{U}_1^k}$ which is imposible because of (star).

Theorem

There is no filter $\mathcal{F}$ such that set $\{0,1\}^\omega/\mathcal{F}$ is countable.

Proof. The set

$$E = \bigcap_{F\in\mathcal{F}}\{\mathcal{U} : \mathcal{U} \text{ is an ultrafilter and } F \in \mathcal{U}\}$$

is closed subset of $\beta\omega \setminus \omega$ and

$$E = \{\mathcal{U} : \mathcal{U} \text{ is ultrafilter and } \mathcal{F} \subset \mathcal{U}\}$$

The set E is infinite: If $E = \{\mathcal{U}_1, ..., \mathcal{U}_n\}$ then for $\mathcal{U} = \bigcap_{i=1}^n \mathcal{U}_i$ and X-selector of $\{0,1\}^\omega/\mathcal{F}$ take point $x \in X \cap [\mathbf{0}]_\mathcal{U} \setminus [\mathbf{0}]_\mathcal{F}$. Then set

$$\{-x\} \cup V_\mathcal{F}$$

is consistent and from previous theorem there exist ultrafilter $\mathcal{U}_{n+1}$ with $\mathcal{F} \subset \mathcal{U}_{n+1}$ and $\{-x\} \cup V_\mathcal{F} \subset [\mathbf{0}]_{\mathcal{U}_{n+1}}$. Then $\mathcal{U}_{n+1} \neq \mathcal{U}_i$ for $i = 1, ..., n$ because of $x \in [\mathbf{0}]_{\mathcal{U}_i}$.

Set $E \subset \beta\omega \setminus \omega$ is closed and infinite thus $|E| = 2^{2^{\aleph_0}}$.
We write $\{0,1\}^\omega/\mathcal{F} = \{X_n : n \in \omega\}$ and construct a function

$$\Phi : E \to 2^\omega$$
$$\Phi(\mathcal{U}) = (\phi_n^{\mathcal{U}})_{n\in\omega} \in 2^\omega$$

where following holds

$$\phi_n^{\mathcal{U}} = 0 \text{ iff } X_n \subset [\mathbf{0}]_{\mathcal{U}}$$
$$\phi_n^{\mathcal{U}} = 1 \text{ iff } X_n \subset [\mathbf{1}]_{\mathcal{U}}$$

We check that Φ is $1-1$ which gives us contradiction $2^{2^{\aleph_0}} \leqslant 2^{\aleph_0}$.

If $\mathcal{U}_0 \neq \mathcal{U}_1$ then $[\mathbf{0}]_{\mathcal{U}_0} \neq [\mathbf{0}]_{\mathcal{U}_0}$ and so there exist $n \in \omega$ with $X_n \subset [\mathbf{0}]_{\mathcal{U}_0}$ and $-X_n \subset [\mathbf{0}]_{\mathcal{U}_1}$ which gives us

$$\phi_n^{\mathcal{U}_0} = 0 \neq 1 = \phi_n^{\mathcal{U}_1}$$

Modification of proof shows that

If $\mathcal{F}$ is such filter that $\{0,1\}^\omega/\mathcal{F}$ has cardinality κ for $\aleph_0 < \kappa < 2^{\aleph_0}$ then

$$2^{2^{\aleph_0}} = 2^\kappa$$

Martin Axiom implies that no such filter exists

THANK YOU