A tall ideal in which player *II* has a winning strategy in the cut and choose game

José de Jesús Pelayo Gómez

Posgrado Conjunto en Ciencias Matemáticas
UNAM-UMSNH
Morelia, México

Winter School in Abstract Analysis
Hejnice, Czech Republic.

Definitions

- Let $\mathcal{I}$ be an ideal on ω. The Cut and Choose Game $G(\mathcal{I})$ is a game between two players.

Definitions

- Let $\mathcal{I}$ be an ideal on ω. The Cut and Choose Game $G(\mathcal{I})$ is a game between two players.
- The player I starts the game by cutting ω into two pieces $\omega = A_0^0 \cup A_1^0$.

Definitions

- Let $\mathcal{I}$ be an ideal on ω. The Cut and Choose Game $G(\mathcal{I})$ is a game between two players.

- The player *I* starts the game by cutting ω into two pieces $\omega = A_0^0 \cup A_1^0$.

- The player *II* chooses one of the pieces and a natural number in that piece. Then player *I* cuts the part that the player *II* chose, etcetera.

Definitions

- Let $\mathcal{I}$ be an ideal on ω. The Cut and Choose Game $G(\mathcal{I})$ is a game between two players.
- The player *I* starts the game by cutting ω into two pieces $\omega = A_0^0 \cup A_1^0$.
- The player *II* chooses one of the pieces and a natural number in that piece. Then player *I* cuts the part that the player *II* chose, etcetera.
- The player *I* wins the game iff the natural numbers that were chosen by player *II* is an element of $\mathcal{I}$.

Definitions

- Let $\mathcal{I}$ be an ideal on ω. The Cut and Choose Game $G(\mathcal{I})$ is a game between two players.
- The player I starts the game by cutting ω into two pieces $\omega = A_0^0 \cup A_1^0$.
- The player II chooses one of the pieces and a natural number in that piece. Then player I cuts the part that the player II chose, etcetera.
- The player I wins the game iff the natural numbers that were chosen by player II is an element of $\mathcal{I}$.

I	$\omega = A_0^0 \cup A_1^0$		$A_{i_0}^0 = A_0^1 \cup A_1^1$		...
II		$i_0 \in 2, n_0 \in A_{i_0}^0$		$i_1 \in 2, n_1 \in A_{i_1}^1$	...

Example

Proposition

The player I has a winning strategy in $G(\mathcal{ED})$.

Example

Proposition

The player I has a winning strategy in $G(\mathcal{ED})$.

Proof

$\mathcal{ED}$ is the ideal on $\omega \times \omega$ generated by the sets of the form $\{n\} \times \omega$ and the graphs of functions.

Example

Proposition

The player I has a winning strategy in $G(\mathcal{ED})$.

Proof

$\mathcal{ED}$ is the ideal on $\omega \times \omega$ generated by the sets of the form $\{n\} \times \omega$ and the graphs of functions. Let $n_{-1} = 0$, the player I plays $A_0 = \{n_{-1}\} \times \omega$ and $B_0 = \{n \in \omega : n > 0\} \times \omega$. If the player II choses A_0 then the player II loses so he must to choose B_0 and $(n_0, m_0) \in B_0$. In every following step the player I plays $A_i = \{n \in \omega : n_{i-2} < n \leq n_{i-1}\} \times \omega$ and $B_i = \{n \in \omega : n > n_{i-1}\} \times \omega$. Always the player II must to choose B_i and so $\{(n_i, m_i) : i \in \omega\}$ is a subset of a graph of a function and the player II will lose the game.

Definitions

- We say that an ideal $\mathcal{I}$ is tall if $\mathcal{I} \upharpoonright A$ is not the Fin ideal for each $A \in [\omega]^\omega$.

Definitions

- We say that an ideal $\mathcal{I}$ is tall if $\mathcal{I} \restriction A$ is not the Fin ideal for each $A \in [\omega]^\omega$.

- We will denote by $\mathcal{R}$ the random graph ideal.

Definitions

- We say that an ideal $\mathcal{I}$ is tall if $\mathcal{I} \restriction A$ is not the Fin ideal for each $A \in [\omega]^\omega$.

- We will denote by $\mathcal{R}$ the random graph ideal.

- We say that $\omega \to (\mathcal{I}^+)^2_2$ if for every coloring (with two colors) of pairs of ω there is a monochromatic $\mathcal{I}$-positive set.

Another example

Proposition

Let $\mathcal{I}$ be a non tall ideal, then the player *II* has a winning strategy in $G(\mathcal{I})$.

Another example

Proposition

Let $\mathcal{I}$ be a non tall ideal, then the player *II* has a winning strategy in $G(\mathcal{I})$.

Proof

If $\mathcal{I}$ is not a tall ideal, then there is $A \in [\omega]^\omega$ such that if $B \in [A]^\omega$ then $B \in \mathcal{I}^+$.

Another example

Proposition

Let $\mathcal{I}$ be a non tall ideal, then the player *II* has a winning strategy in $G(\mathcal{I})$.

Proof

If $\mathcal{I}$ is not a tall ideal, then there is $A \in [\omega]^\omega$ such that if $B \in [A]^\omega$ then $B \in \mathcal{I}^+$. When the player *I* cuts ω (or the part that the player *II* chooses) one of the parts has an infinite intersection with A

Another example

Proposition

Let $\mathcal{I}$ be a non tall ideal, then the player *II* has a winning strategy in $G(\mathcal{I})$.

Proof

If $\mathcal{I}$ is not a tall ideal, then there is $A \in [\omega]^\omega$ such that if $B \in [A]^\omega$ then $B \in \mathcal{I}^+$. When the player *I* cuts ω (or the part that the player *II* chooses) one of the parts has an infinite intersection with A so the player *II* chooses that piece and some natural number in A. At the end, the player *I* loses the game because the natural numbers chosen by the player *II* form an infinite subset of A.

Examples and observations

- The player I has a winning strategy in $G(\mathcal{R})$.

Examples and observations

- The player I has a winning strategy in $G(\mathcal{R})$.
- If the player I has a winning strategy in $G(\mathcal{I})$ and $\mathcal{I} \leq_K \mathcal{J}$ then the player I has a winning strategy in $G(\mathcal{J})$.

Examples and observations

- The player I has a winning strategy in $G(\mathcal{R})$.
- If the player I has a winning strategy in $G(\mathcal{I})$ and $\mathcal{I} \leq_K \mathcal{J}$ then the player I has a winning strategy in $G(\mathcal{J})$.
- If the player II has a winning strategy in $G(\mathcal{I})$ then $\omega \to (\mathcal{I}^+)^2_2$.

Examples and observations

- The player I has a winning strategy in $G(\mathcal{R})$.
- If the player I has a winning strategy in $G(\mathcal{I})$ and $\mathcal{I} \leq_K \mathcal{J}$ then the player I has a winning strategy in $G(\mathcal{J})$.
- If the player II has a winning strategy in $G(\mathcal{I})$ then $\omega \to (\mathcal{I}^+)^2_2$.
- If $\mathcal{I}$ is a non tall ideal then the player II has a winning strategy in $G(\mathcal{I})$.

Questions

Recall that $\omega \to (\mathcal{I}^+)^2_2$ iff $\mathcal{R} \not\leq_K \mathcal{I}$.

Questions

Recall that $\omega \to (\mathcal{I}^+)^2_2$ iff $\mathcal{R} \not\leq_K \mathcal{I}$.

Question (J. Zapletal)

Does the player *II* have a winning strategy in $G(\mathcal{S})$?

Questions

Recall that $\omega \to (\mathcal{I}^+)^2_2$ iff $\mathcal{R} \not\leq_K \mathcal{I}$.

Question (J. Zapletal)

Does the player *II* have a winning strategy in $G(\mathcal{S})$?

Question (M. Hrusak)

Does $\mathcal{S}$ satisfy $\omega \to (\mathcal{S}^+)^2_2$?

Questions

Recall that $\omega \to (\mathcal{I}^+)^2_2$ iff $\mathcal{R} \not\leq_K \mathcal{I}$.

Question (J. Zapletal)

Does the player *II* have a winning strategy in $G(\mathcal{S})$?

Question (M. Hrusak)

Does $\mathcal{S}$ satisfy $\omega \to (\mathcal{S}^+)^2_2$?

Question (J. Zapletal)

If the player *II* has a winning strategy in $G(\mathcal{I})$ then $\mathcal{I}$ is not tall?

Questions

Proposition

$\omega \to (\omega, \mathcal{S}^+)^2_2$.

More definitions

We will define a tall F_σ ideal in which the player II has a winning strategy in the cut and choose game. Let $\{A_s : s \in \omega^{<\omega}\}$ be a family of subsets of ω such that:

More definitions

We will define a tall F_σ ideal in which the player II has a winning strategy in the cut and choose game. Let $\{A_s : s \in \omega^{<\omega}\}$ be a family of subsets of ω such that:

- $A_\emptyset = \omega$
- For every $s \in \omega^{<\omega}$ we have that $\{A_{s^\frown n} : n \in \omega\}$ is a partition of A_s.
- For every $n \neq m$ natural numbers $\exists s \neq t \in \omega^{<\omega}$ such that $n \in A_s$ and $m \in A_t$.

We will say that $T \subseteq \omega^{<\omega}$ is a small-branching tree if for every $s \in T$

$$|\{n \in \omega : s^\frown n \in T\}| \leq |s| + 1.$$

With the previous notation, we define

$$\mathcal{S}_0 = \{A \subseteq \omega : \exists s \in \omega^{<\omega}(A \subseteq A_s \wedge \forall n \in \omega(|A \cap A_{s^\frown n}| = 1))\} \text{ and}$$

$$\mathcal{S}_1 = \{A \subseteq \omega : \{s \in \omega^{<\omega} : A_s \cap A \neq \emptyset\} \text{ is a small-branching tree}\}.$$

$\mathcal{PC}$ is the ideal generated by $\mathcal{S}_0 \cup \mathcal{S}_1$.

Main result

Theorem

$\mathcal{PC}$ is a tall F_σ ideal and the player II has a winnning strategy in $G(\mathcal{PC})$.

Main result

Theorem

$\mathcal{PC}$ is a tall F_σ ideal and the player *II* has a winnning strategy in $G(\mathcal{PC})$.

Sketch of proof

After the player *I* cuts ω or the part which was chosen by the player *II* in a previous step, the player *II* chooses the big part (big in some way) and a certain natural number. The strategy is that the player *II* must construct some set such that the corresponding tree is not the finite union of small braching trees or trees corresponding to selectors.

GRACIAS
THANK YOU
DEKUJI.