The ultrafilter number for uncountable κ

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Section 1

The ultrafilter number on ω.

The ultrafilter number and its neighbors.

We will focus our interest in the cardinal invariant defined by:

Definition

$\mathfrak{u} = \min\{|\mathcal{B}|: \mathcal{B}$ *is a base for a non-principal ultrafilter on* $\omega\}$

It is ZFC provable that (Blass- Combinatorial cardinal characteristics of the continuum):

- $\aleph_1 \leq \mathfrak{u} \leq \mathfrak{c}$.
- One of its lowers bounds is the cardinal $\mathfrak{r}$, and as consequence $\mathfrak{b}$, $\mathfrak{e}$, $\mathfrak{h}$, $\mathfrak{t}$ and $\mathfrak{p}$.

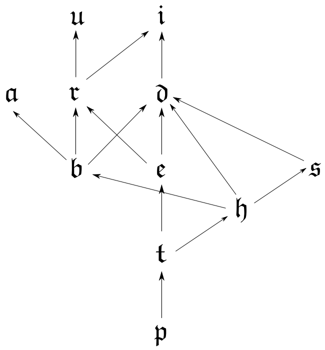

Figure 1: $\mathfrak{u}$ and its neighbors

Using forcing it is possible to prove, that:

Theorem (Kunen, Lemma V.4.27 in [5])

It is consistent that $\mathfrak{u} = \aleph_1$ *and* $\mathfrak{c} = \kappa$ *for* $\kappa > \aleph_1$.

How?... We use Mathias forcing:

Definition (Ultrafilter Mathias Forcing)

Let $\mathcal{U}$ *be an ultrafilter on* ω. *The ultrafilter Mathias forcing* $\mathbb{M}_{\mathcal{U}}$ *has, as its set of conditions,* $\{(s, A) : s \in [\omega]^{<\omega}$ *and* $A \in \mathcal{U}\}$, *and the ordering given by:*
$(t, B) \leq (s, A)$ *if and only if* $t \supseteq s, B \subseteq A$ *and* $t \setminus s \subseteq A$.

Idea of the proof:

Start with a ground model in which $\mathfrak{c} = \kappa$, the forcing is obtained as a finite support iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \aleph_1, \beta < \aleph_1)$ of Mathias forcing relative to some ultrafilters that are constructed along the iteration.

Remember that, if at step $\alpha < \omega_1$ Mathias forcing respect to an ultrafilter $\mathcal{U}_\alpha$ in $V^{\mathbb{P}_\alpha}$ is used (let $\dot{\mathcal{U}}_\alpha$ be a $\mathbb{P}_\alpha$-name for it), we add generically a subset of ω, X_α that is a pseudointersection of $\dot{\mathcal{U}}_\alpha$, i.e. in $V^{\mathbb{P}}$ we have that, for all $F \in \mathcal{U}_\alpha$, $X_\alpha \subseteq^* F$.

Thus, define $\mathbb{P}_0 = \mathbb{1}$ and $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{M}(\dot{\mathcal{U}}_\alpha)$ where $\dot{\mathcal{U}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a non principal ultrafilter that satisfies both $X_\alpha \in \dot{\mathcal{U}}_{\alpha+1}$ and $\forall \beta < \alpha$, $\Vdash_\alpha \dot{\mathcal{U}}_\beta \subseteq \dot{\mathcal{U}}_\alpha$.

Finally, the inequality $\mathfrak{u} \leq \aleph_1$ will be witnessed by the ultrafilter $\bigcup_{\alpha<\omega_1} \dot{\mathcal{U}}_\alpha$ generated by the sets $\{X_\alpha : \alpha < \aleph_1\}$; and the ccc will guarantee that $\mathfrak{c} = \kappa$ still holds in $V^{\mathbb{P}}$.

Section 2

The uncountable case

In this talk we will be mainly interested in the generalization of the ultrafilter number and the analogue of the result presented in the section before ($Con(\mathfrak{u} < \mathfrak{c})$) for an uncountable cardinal κ.

Definition

$\mathfrak{u}(\kappa) = \min\{|\mathcal{B}|$: $\mathcal{B}$ *is an base for a uniform ultrafilter on* $\kappa\}$.
Uniform just means that all the sets in the ultrafilter have size κ.

Subsection 1

Our result

Theorem

Suppose κ is a supercompact cardinal, κ^* is a regular cardinal with $\kappa < \kappa^* \leq \Gamma$ and Γ is a cardinal that satisfies $\Gamma^\kappa = \Gamma$. Then there is a forcing extension in which cardinals have not been changed satisfying:

$$\kappa^* = \mathfrak{u}(\kappa) = \mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \mathfrak{a}(\kappa) = \mathfrak{s}(\kappa) = \mathfrak{r}(\kappa) = \mathrm{cov}(\mathcal{M}_\kappa) \\ = \mathrm{add}(\mathcal{M}_\kappa) = \mathrm{non}(\mathcal{M}_\kappa) = \mathrm{cof}(\mathcal{M}_\kappa) \text{ and } 2^\kappa = \Gamma.$$

If in addition $\gamma < \kappa^* \to \gamma^{<\kappa} < \kappa^*$, then we can also provide that $\mathfrak{i}(\kappa) = \kappa^*$. If in addition $(\Gamma)^{<\kappa^*} \leq \Gamma$ then we can also provide that $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa) = \mathfrak{h}_{\mathcal{W}}(\kappa) = \kappa^*$ where $\mathcal{W}$ is a κ-complete ultrafilter on κ.

Theorem

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Generalized Mathias

First we explain how to build a model where $\mathfrak{u}(\kappa) = \kappa^* > \kappa^+$ and $2^\kappa = \Gamma$. Again we will use a generalized version of Mathias forcing, namely:

Definition (Generalized Mathias Forcing)

Let κ be a measurable cardinal, and let $\mathcal{F}$ be a κ-complete filter on κ. The Generalized Mathias Forcing $\mathbb{M}^\kappa_\mathcal{F}$ has, as its set of conditions, $\{(s, A) : s \in [\kappa]^{<\kappa}$ and $A \in \mathcal{F}\}$, and the ordering given by $(t, B) \leq (s, A)$ if and only if $t \supseteq s, B \subseteq A$ and $t \setminus s \subseteq A$. We denote by $\mathbb{1}_\mathcal{F}$ the maximum element of $\mathbb{M}^\kappa_\mathcal{F}$, that is $\mathbb{1}_\mathcal{F} = (\emptyset, \kappa)$.

The main model

In the uncountable case, it is necessary to use a more sophisticated forcing notion. The reason for which the standard generalization of the proof for the countable case (i.e. a $< \kappa$-support iteration of generalized Mathias forcing) does not work is the following:

If $(\mathcal{U}_n : n \in \omega)$ is an increasing sequence of κ-complete ultrafilters it is possible that $\bigcup_{n\in\omega} \mathcal{U}_n$ is not even a κ-complete filter. ☹ ☺.

Let Γ be such that $\Gamma^\kappa = \Gamma$. We will define an iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \Gamma^+, \beta < \Gamma^+ \rangle$ of length Γ^+ recursively as follows:

Suppose we are in the model $V^{\mathbb{P}_\alpha}$ and let NUF denote the set of normal ultrafilters on κ in this model.

- If α is an even ordinal, let $\mathbb{Q}_\alpha = \{\mathbb{1}_{\mathbb{Q}_\alpha}\} \cup \{\{\mathcal{U}\} \times \mathbb{M}^\kappa_\mathcal{U} : \mathcal{U} \in \text{NUF}\}$ and extension relation stating that $q \leq p$ if and only if either $p = \mathbb{1}_{\mathbb{Q}_\alpha}$, or there is $\mathcal{U} \in \text{NUF}$ such that $p = (\mathcal{U}, p_1)$, $q = (\mathcal{U}, q_1)$ and $q_1 \leq_{\mathbb{M}^\kappa_\mathcal{U}} p_1$.
- If α is an odd ordinal, let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for a κ-centered, κ-directed closed forcing notion of size at most Γ.

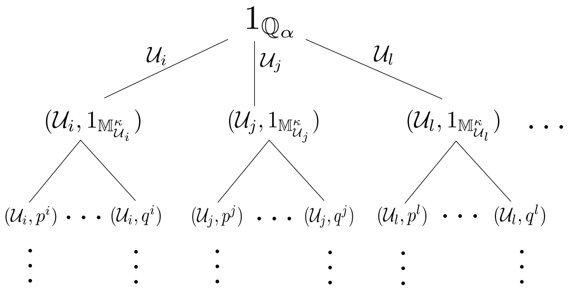

Figure 2: The forcing $\mathbb{Q}_\alpha$ (α even) in the $V^{\mathbb{P}_\alpha}$ extension.

Given a condition $p \in \mathbb{P} = \mathbb{P}_{\Gamma^+}$, we will have three kinds of support:

- The Ultrafilter Support ($\mathrm{USup}(p)$), that corresponds to the set of ordinals $\beta \in \mathrm{dom}(p) \cap \mathrm{EVEN}$ such that $p \restriction \beta \Vdash_{\mathbb{P}_\beta} p(\beta) \neq \mathbb{1}_{\mathbb{Q}_\beta}$.
- Then Essential Support ($\mathrm{SSup}(p)$), which consists of all $\beta \in \mathrm{dom}(p) \cap \mathrm{EVEN}$ such that $\neg(p \restriction \beta \Vdash_{\mathbb{P}_\beta} p(\beta) \in \{\check{\mathbb{1}}_{\mathbb{Q}_\beta}\} \cup \{(\mathcal{U}, \mathbb{1}_{\mathcal{U}}) : \mathcal{U} \in \mathrm{NUF}\})$.
- The Directed Support $\mathrm{RSup}(p)$, consists of all $\beta \in \mathrm{dom}(p) \cap \mathrm{ODD}$ such that $\neg(p \restriction \beta \Vdash p(\beta) = \mathbb{1}_{\dot{\mathbb{Q}}_\beta})$.

Then we require that the conditions in $\mathbb{P}_{\Gamma^+}$ have support bounded below Γ^+ and also that given $p \in \mathbb{P}_{\Gamma^+}$, if $\beta \in \mathrm{USup}(p)$ then for all $\alpha \in \beta$, $\alpha \in \mathrm{USup}(p)$ and that both $\mathrm{SSup}(p)$ and $\mathrm{RSup}(P)$ have size $< \kappa$ and are contained in $\sup(\mathrm{USup}(p))$.

Properties of the forcing $\mathbb{P} = \mathbb{P}_{\Gamma^+}$:

- $\mathbb{P}$ is κ-directed closed.
- If $p \in \mathbb{P}_{\Gamma^+}$ and $i = \sup(\mathrm{USup}(p)) = \sup(\mathrm{supp}(p))$. Then $\mathbb{P}_i \downarrow (p \restriction i)$ is κ^+-cc and has a dense subset of size at most Γ.
- **The key property:** Suppose that $p \in \mathbb{P}$ is such that $p \Vdash \dot{\mathcal{U}}$ is a normal ultrafilter on κ, then for some $\alpha < \Gamma^+$ there is an extension $q \leq p$ such that $q \Vdash \dot{\mathcal{U}}_\alpha = \dot{\mathcal{U}} \cap V[G_\alpha]$. Moreover this can be done for a set of ordinals $S \subseteq \Gamma^+$ of order type κ^* in such a way that $\forall \alpha \in S(\dot{\mathcal{U}} \cap V_\alpha \in V[G_\alpha])$ and $\dot{\mathcal{U}} \cap V[G_{\sup S}] \in V[G_{\sup S}]$. Here $\dot{\mathcal{U}}_\alpha$ is the canonical name for the ultrafilter generically chosen at stage α.

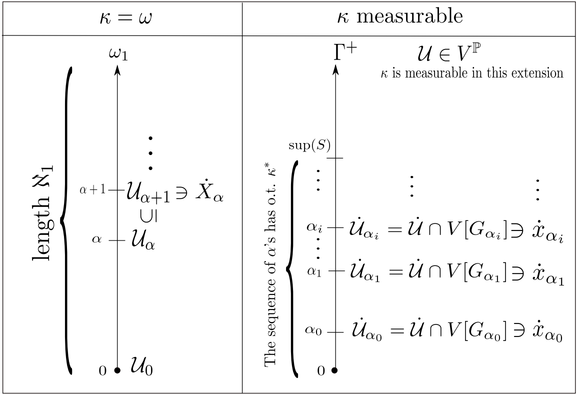

Figure 3: Methods to find an ultrafilter with a small base.

Theorem

Suppose κ is a supercompact cardinal and κ^* is a regular cardinal with $\kappa < \kappa^* \leq \Gamma$, $\Gamma^\kappa = \Gamma$. There is a forcing notion $\mathbb{P}^*$ preserving cofinalities such that $V^{\mathbb{P}^*} \models \mathfrak{u}(\kappa) = \kappa^* \wedge 2^\kappa = \Gamma$.

Proof idea: We will not work with the whole generic extension given by $\mathbb{P}$. In fact we will chop the iteration in the step $\alpha = \sup(S)$ which is an ordinal of cofinality κ^*. Define $\mathbb{P}^* = \mathbb{P}_\alpha$.

Take G to be a $\mathbb{P}^*$-generic filter, the equality $2^\kappa = \Gamma$ is a consequence of the fact that the domains of the conditions obtained in the key property can be chosen in such a way that they all have size Γ.

To prove $\mathfrak{u}(\kappa) = \kappa^*$ we consider the ultrafilter $\mathcal{U}^*$ on κ given by the restriction of $\mathcal{U}$. Then by the same lemma note that for all $i \in S$ the restriction of $\mathcal{U}$ to the model $V[G_i]$ belongs to $V[G_{i+1}]$ and moreover, this is the ultrafilter U_i^G chosen generically at stage i.

Furthermore by our choice of Master Conditions the κ-Mathias generics $\dot{x}_i$ belong to $\mathcal{U}$. Then $\mathcal{U}^*$ is generated by $\dot{x}_i$ for $i \in S$. The other inequality $\mathfrak{u}(\kappa) \geq \kappa^*$ is a consequence of $\mathfrak{b}(\kappa) \geq \kappa^*$ and $\mathfrak{b}(\kappa) \leq \mathfrak{u}(\kappa)$.

Section 3

Applications

Other cardinal invariants that can be decided in $V^{\mathbb{P}}$

Definition

The unbounding and dominating numbers, $\mathfrak{b}(\kappa)$ *and* $\mathfrak{d}(\kappa)$ *respectively are defined as follows:*

$$\mathfrak{b}(\kappa) = \min\{|\mathfrak{F}| \colon \mathfrak{F} \text{ is an unbounded family of functions from } \kappa \text{ to } \kappa\}.$$

$$\mathfrak{d}(\kappa) = \min\{|\mathfrak{F}| \colon \mathfrak{F} \text{ is a dominating family of functions from } \kappa \text{ to } \kappa\}.$$

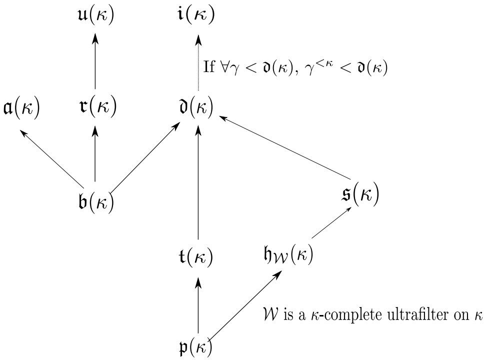

Figure 4: Provable inequalities for κ-measurable.

Definition (Generalized Laver forcing)

Let $\mathcal{U}$ be a κ-complete non-principal ultrafilter on κ.

- *A $\mathcal{U}$-Laver tree is a κ-closed tree $T \subseteq \kappa^{<\kappa}$ of increasing sequences with the property that $\forall s \in T(|s| \geq |\mathrm{stem}(T)| \rightarrow \mathrm{succ}_T(s) \in \mathcal{U})\}$.*
- *The generalized Laver Forcing $\mathbb{L}^{\kappa}_{\mathcal{U}}$ consists of all $\mathcal{U}$-Laver trees with order given by inclusion.*

Proposition

Generalized Laver forcing $\mathbb{L}^{\kappa}_{\mathcal{U}}$ generically adds a dominating function from κ to κ.

Lemma

If $\mathcal{U}$ is a normal ultrafilter on κ, then $\mathbb{M}^{\kappa}_{\mathcal{U}}$ and $\mathbb{L}^{\kappa}_{\mathcal{U}}$ are forcing equivalent.

Corollary

If $\mathcal{U}$ is a normal ultrafilter on κ then $\mathbb{M}^{\kappa}_{\mathcal{U}}$ always adds dominating functions, so we have $\mathfrak{b}(\kappa) = \kappa^* = \mathfrak{d}(\kappa)$.

Proposition

In $V^{\mathbb{P}}$, $\mathfrak{s}(\kappa)$ and $\mathfrak{r}(\kappa)$ also take the value κ^*.

The intermediate forcing

Until now, we have not used the poset added in the odd steps of our iteration, we will do it in order to decide the cardinal characteristics $\mathfrak{i}(\kappa)$ and $\mathfrak{a}(\kappa)$ in the resulting model. Remember that in these steps the forcing takes a name for an arbitrary κ-centered, κ-directed closed forcing notion of size at most Γ.

We will prove that both cardinals $\mathfrak{a}(\kappa)$ and $\mathfrak{i}(\kappa)$ take also the value κ^* by introducing (with the help of the odd steps) witnesses for $\mathfrak{a}(\kappa), \mathfrak{i}(\kappa) \leq \kappa^*$.

Definition

Two sets A and $B \in \mathcal{P}(\kappa)$ are called κ-almost disjoint if $A \cap B$ has size $< \kappa$. We say that a family of sets $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ is κ-almost disjoint if it has size at least κ and all its elements are pairwise κ-almost disjoint. A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is called a κ-maximal almost disjoint (abbreviated κ-mad) if it is κ-almost disjoint and is not properly included in another such family.

Definition

$\mathfrak{a}(\kappa) = \min\{|\mathfrak{A}| : \mathfrak{A} \text{ is a } \kappa\text{-mad family}\}$.

Poset adding a κ-mad family

Definition

Let $\mathcal{A} = \{A_i\}_{i<\delta}$ be a κ-almost disjoint family. Let $\bar{\mathbb{Q}}(\mathcal{A}, \kappa)$ be the poset of all pairs (s, F) where $s \in 2^{<\kappa}$ and $F \in [\mathcal{A}]^{<\kappa}$, with extension relation stating that $(t, H) \leq (s, F)$ if and only if $t \supseteq s$, $H \supseteq F$ and for all $i \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$ with $t(i) = 1$ we have $i \notin \bigcup\{A : A \in F\}$.

Note that the poset $\bar{\mathbb{Q}}(\mathcal{A}, \kappa)$ is κ-centered and κ-directed closed. If G is $\bar{\mathbb{Q}}(\mathcal{A}, \kappa)$-generic then $\chi_G = \bigcup\{t : \exists F(t, F) \in G\}$ is the characteristic function of an unbounded subset x_G of κ such that $\forall A \in \mathcal{A}(|A \cap x_G|) < \kappa$.

The poset $\mathbb{Q}$ has the following property, that in fact ensures maximality.

Proposition

If $Y \in [\kappa]^\kappa \setminus \mathcal{I}_\mathcal{A}$, where $\mathcal{I}_\mathcal{A}$ is the κ-complete ideal generated by the κ-ad-family $\mathcal{A}$, then $\Vdash_{\Theta(\mathcal{A},\kappa)} |Y \cap \dot{x}_G| = \kappa$.

Note: The construction for the witnesses of $\mathfrak{i}(\kappa)$ is similar to the one we just present for the case of mad families.

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