Mathias forcing for filters and combinatorial covering properties

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Mathias forcing for filters

A subset $\mathcal{F}$ of $[\omega]^\omega$ is called a *filter* if $\mathcal{F}$ contains all cofinite sets, is closed under finite intersections of its elements, and under taking supersets.

$\mathbb{M}_\mathcal{F}$ consists of pairs $\langle s, F\rangle$ such that $s \in [\omega]^{<\omega}$, $F \in \mathcal{F}$, and $\max s < \min F$. A condition $\langle s, F\rangle$ is stronger than $\langle t, U\rangle$ if $F \subset U$, s is an end-extension of t, and $s \setminus t \subset U$.

$\mathbb{M}_\mathcal{F}$ is usually called *Mathias forcing associated with* $\mathcal{F}$.

$\mathbb{M}_\mathcal{F}$ is a natural forcing adding a pseudointersection of $\mathcal{F}$: if G is a $\mathbb{M}_\mathcal{F}$-generic, then $X = \bigcup\{s : \exists F \in \mathcal{F}(\langle s, F\rangle \in G)\}$ is almost contained in any $F \in \mathcal{F}$.

Applications: killing mad families, making the ground model reals not splitting, etc.

$\mathbb{M}_{\mathcal{F}}$ and dominating reals

Let $x, y \in \omega^\omega$. The notation $x \leq^* y$ means $x(n) \leq y(y)$ for all but finitely many n.

$\mathfrak{b}$ (resp. $\mathfrak{d}$) is the minimal size of an $\leq^*$-unbounded (resp. dominating) $A \subset \omega^\omega$.

A poset $\mathbb{P}$ is said to *add a dominating real* if in $V^{\mathbb{P}}$ there exists $x \in \omega^\omega$ such that $y \leq^* x$ for all ground model $y \in \omega^\omega$. .

Example: Laver forcing, Hechler forcing.

Miller and Cohen forcing do not add dominating reals.

Theorem (Canjar 1988)

$\mathfrak{d} = \mathfrak{c}$ *implies the existence of an ultrafilter* $\mathcal{F}$ *such that* $\mathbb{M}_{\mathcal{F}}$ *does not add dominating reals.* □

Definition (Guzman-Hrusak-Martinez)

A filter $\mathcal{F}$ *on* ω *is called Canjar if* $\mathbb{M}_{\mathcal{F}}$ *does not add dominating reals.*
Let B *be an unbounded subset of* ω^ω. *A filter* $\mathcal{F}$ *on* ω *is called* B*-Canjar if* $\mathbb{M}_{\mathcal{F}}$ *adds no reals dominating all elements of* B. □

There is a combinatorial characterization of Canjar filters by Hrusak and Minami in terms of the filter $\mathcal{F}^{<\omega}$ on $[\omega]^{<\omega}$ generated by $\{[F]^{<\omega} : F \in \mathcal{F}\}$.

$\mathbb{M}_{\mathcal{F}}$ and dominating reals: continuation

Theorem (Brendle 1998)

1) Every σ-compact filter is Canjar.
2) $(\mathfrak{b} = \mathfrak{c})$. Let $\mathcal{A}$ be a mad family. Then for any unbounded $B = \{b_\alpha : \alpha < \mathfrak{b}\} \subset \omega^\omega$ such that $b_\alpha \leq^* b_\beta$ for all $\alpha < \beta$, there exists a B-Canjar $\mathcal{F} \supset \mathcal{F}_{\mathcal{A}}$. □

If an ultrafilter $\mathcal{F}$ is Canjar, then it is a P-filter and there is no monotone surjection $\varphi : \omega \to \omega$ such that $\varphi(\mathcal{F})$ is rapid. The converse is consistently not true by a recent result of Blass, Hrusak and Verner. Its proof relies on the following characterization

Theorem (Guzman-Hrusak-Martinez 2013; Blass-Hrusak-Verner 2011 for ultrafilters)

A filter $\mathcal{F}$ is Canjar iff it is a coherent strong P^+-filter. □

Recall that a filter $\mathcal{F}$ is a *coherent strong P^+-filter* if for every sequence $\langle \mathcal{C}_n : n \in \omega \rangle$ of compact subsets of $\mathcal{F}^+$ there exists an increasing sequence $\langle k_n : n \in \omega \rangle$ of integers such that if $X_n \in \mathcal{C}_n$ for all n and $X_m \cap [k_n, k_{n+1}) \subset X_n \cap [k_n, k_{n+1})$ for $n < m$, then $\bigcup_{n\in\omega}(X_n \cap [k_n, k_{n+1})) \in \mathcal{F}^+$.
Strong P^+-filters are defined by removing the coherence requirement.

Covering properties of Menger and Hurewicz

A topological space X has the Menger covering property (or simply is Menger), if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{\bigcup \mathcal{V}_n : n \in \omega\}$ is a cover of X.

If, moreover, we can choose $\mathcal{V}_n$ in such a way that for any $x \in X$ we have $x \in \bigcup \mathcal{V}_n$ for all but finitely many $n \in \omega$, then X is called Hurewicz. □

Example: every σ-compact space is Hurewicz. More generally: a union of fewer than $\mathfrak{b}$ (resp. $\mathfrak{d}$) compacts is Hurewicz (resp. Menger).

ω^ω is not Menger as witnessed by $\langle \mathcal{U}_n : n \in \omega \rangle$,
$\mathcal{U}_n = \big\{\{x : x(n) = k\} \ : \ k \in \omega\big\}$.

Main results

Theorem (Chodounský-Repovš-Z. 2014)

$\mathbb{M}_{\mathcal{F}}$ is Canjar iff $\mathcal{F}$ has the Menger covering property as a subspace of $\mathcal{P}(\omega)$. □

Theorem (Chodounský-Repovš-Z. 2014)

Let $\mathcal{F}$ be a filter. Then $\mathbb{M}_{\mathcal{F}}$ is almost ω^ω-bounding iff $\mathcal{F}$ is B-Canjar for all unbounded $B \subset \omega^\omega$ iff $\mathcal{F}$ is Hurewicz. □

Recall that a poset $\mathbb{P}$ is almost ω^ω-bounding if for every $\mathbb{P}$-name $\dot{f}$ for a real and $q \in \mathbb{P}$, there exists $g \in \omega^\omega$ such that for every $A \in [\omega]^\omega$ there is $q_A \leq q$ such that $q_A \Vdash g \restriction A \not\leq^* \dot{f} \restriction A$.

Some corollaries

Corollary

Let $\mathcal{F}$ be an analytic filter on ω. Then $\mathbb{M}_{\mathcal{F}}$ does not add a dominating real iff $\mathcal{F}$ is σ-compact. $\square$

Answers a question of Hrusak and Minami. For Borel filters has been independently proved by Guzman, Hrusak, and Martinez.

Corollary (Hrušák-Martínez 2012)

There exists a mad family $\mathcal{A}$ on ω such that $\mathbb{M}_{\mathcal{F}(\mathcal{A})}$ adds a dominating real (= $\mathcal{F}(\mathcal{A})$ is not Canjar). $\square$

Answers a question of Brendle.

Corollary

($\mathfrak{d} = \mathfrak{c}$.) There exists a mad family $\mathcal{A}$ on ω such that $\mathbb{M}_{\mathcal{F}(\mathcal{A})}$ does not add a dominating real (= $\mathcal{F}$ is Canjar). $\square$

Under $\mathfrak{d} = \mathfrak{c} = \mathfrak{u}$ it was proved by Guzman, Hrusak, and Martinez.

Corollary

A filter $\mathcal{F}$ is Canjar iff it is a strong P^+-filter. $\square$

Theorem (Guzman-Hrusak-Martinez 2013)

A filter $\mathcal{F}$ is Canjar iff it is a coherent strong P^+-filter. □

Recall that a filter $\mathcal{F}$ is a *coherent strong P^+-filter* if for every sequence $\langle \mathcal{C}_n \colon n \in \omega \rangle$ of compact subsets of $\mathcal{F}^+$ there exists an increasing sequence $\langle k_n \colon n \in \omega \rangle$ of integers such that if $X_n \in \mathcal{C}_n$ for all n
and $X_m \cap [k_n, k_{n+1}) \subset X_n \cap [k_n, k_{n+1})$ for $n < m$,
then $\bigcup_{n\in\omega}(X_n \cap [k_n, k_{n+1})) \in \mathcal{F}^+$.

Strong P^+-filters are defined by removing the coherence requirement.

An auxiliary claim.

For $n \in \omega$ and $q \subset n$ we set $[n, q] := \{A \in \mathcal{P}(\omega) \colon A \cap n = q\}$. Sets $[n, q]$ form a standard base $\mathcal{B}$ for the topology of $\mathcal{P}(\omega)$. Set also $\uparrow X = \{A \in \mathcal{P}(\omega) \colon A \supset X\}$ for every $X \subset \omega$.

Lemma

Suppose that $\mathcal{X} \subset \mathcal{P}(\omega)$ is closed under taking supersets and $\mathcal{O}$ is a cover of $\mathcal{X}$ by sets open in $\mathcal{P}(\omega)$. Then there exists a family $Q \subset [\omega]^{<\omega}$ such that $\mathcal{X} \subset \bigcup_{q \in Q} \uparrow q$ and for every $q \in Q$ there exists $\mathcal{O}' \in [\mathcal{O}]^{<\omega}$ covering $\uparrow q$.

Proof. Wlog $\mathcal{O} \subset \mathcal{B}$. Let us fix $X \in \mathcal{X}$ and find $\{[n_i, q_i] \colon i \in m\} \subset \mathcal{O}$ such that $\uparrow X \subset \bigcup_{i \in m} [n_i, q_i]$. Breaking some of the sets $[n_i, q_i]$ into smaller pieces of the same form, we may assume if necessary that for some $n \in \omega$ we have $n_i = n$ for all $i \in m$. Moreover, wlog no proper subcollection of $\mathcal{O}' = \{[n, q_i] \colon i < m\}$ covers $\uparrow X$. Therefore $\{q_i \colon i < m\} = \{t \subset n \colon X \cap n \subset t\}$, and consequently $\bigcup_{i<m} [n, q_i] = \uparrow (X \cap n)$. Thus $X \in \uparrow X \subset \uparrow (X \cap n) \subset \bigcup \mathcal{O}'$. $\square$

Proof of "$\mathcal{F}$ is Hurewicz iff $\mathbb{M}_{\mathcal{F}}$ is almost ω^ω-bounding".

Suppose that $\mathcal{F}$ is Hurewicz, but there exists an unbounded $X \subset \omega^\omega$, $X \in V$, and an $\mathbb{M}_{\mathcal{F}}$-name $\dot{g}$ for a function dominating X (as forced by $1_{\mathbb{M}_{\mathcal{F}}}$). For every $x \in X$ find $n^x \in \omega$ and a condition $\langle s^x, F^x \rangle$ forcing $x(n) < \dot{g}(n)$ for all $n \geq n^x$. Since X cannot be covered by a countable family of bounded sets, wlog $s^x = s_*$ and $n^x = n_*$ for all $x \in X$.

For every $m \in \omega$ consider
$\mathcal{S}_m = \{s \in [\omega]^{<\omega} : \max s_* < \min s \ \wedge \ \exists F_s \in \mathcal{F} \, (\langle s_* \cup s, F_s \rangle \Vdash \dot{g}(m) = g_s(m))\}$.
For every $F \in \mathcal{F}$ there exists $s \in \mathcal{S}_m$ such that $s \subset F$. In other words, $\mathcal{U}_m := \{\uparrow s \colon s \in \mathcal{S}_m\}$ is an open cover of $\mathcal{F}$. Since $\mathcal{F}$ is Hurewicz, for every m there exists $\mathcal{V}_m \in [\mathcal{U}_m]^{<\omega}$ such that $\{\bigcup \mathcal{V}_m \colon m \in \omega\}$ is a γ-cover of $\mathcal{F}$. Let $\mathcal{T}_m \in [\mathcal{S}_m]^{<\omega}$ be such that $\mathcal{V}_m = \{\uparrow s \colon s \in \mathcal{T}_m\}$ and $f(m) = \max\{g_s(m) \colon s \in \mathcal{T}_m\}$. We will derive a contradiction by showing $x <^* f$ for each $x \in X$.

Fix $x \in X$ and $l \in \omega$ such that for every $m \geq l$ there exists $s_m \in \mathcal{T}_m$ such that $F^x \in\uparrow s_m$. Pick any $m \geq n_*, l$. Since $\langle s_*, F^x \rangle \Vdash x(m) < \dot{g}(m)$, $\langle s_* \cup s_m, F_{s_m} \rangle \Vdash \dot{g}(m) \leq f(m)$, and these two conditions are compatible, it follows that $x(m) < f(m)$.

Now suppose that $\mathcal{F}$ is not Hurewicz as witnessed by a sequence $\langle \mathcal{U}_n \colon n \in \omega \rangle$ of covers of $\mathcal{F}$ by sets open in $\mathcal{P}(\omega)$. Wlog $\mathcal{U}_n = \{\uparrow q_m(n) \colon m \in \omega\}$, where $q_m(n) \in [\omega]^{<\omega}$. For every $F \in \mathcal{F}$ consider the function $x_F \in \omega^\omega$, $x_F(n) = \min\{m \colon F \in\uparrow q_m(n)\}$. $X = \{x_F \colon F \in \mathcal{F}\}$ is unbounded.

Let G be the generic pseudointersection of $\mathcal{F}$ added by $\mathbb{M}_{\mathcal{F}}$. For every n there exists $g(n)$ such that $G \setminus n \in\uparrow q_{g(n)}(n)$. Fix $F \in \mathcal{F}$ and find n such that $G \setminus n \subset F$. Then $G \setminus n \in\uparrow q_{g(n)}(n)$ yields $F \in\uparrow q_{g(n)}(n)$, which implies $x_F(n) \leq g(n)$. Thus $g \in \omega^\omega$ is dominating X, and therefore $\mathbb{M}_{\mathcal{F}}$ fails to preserve ground model unbounded sets. $\square$

Questions

Question

Let $\mathcal{A} \subset [\omega]^\omega$ be a mad family. Is there a Hurewicz filter $\mathcal{F}$ containing $\mathcal{F}(\mathcal{A})$?

Question

(CH) Let $\mathcal{U}$ be a meager filter generated by a tower. Is there a Hurewicz filter $\mathcal{F}$ containing $\mathcal{U}$?

The last slide

Thank you for your attention.