The structure of Fréchet-Urysohn and radial spaces

Robert Leek
DPhil student, University of Oxford
robert.leek@maths.ox.ac.uk
www.maths.ox.ac.uk/people/profiles/robert.leek

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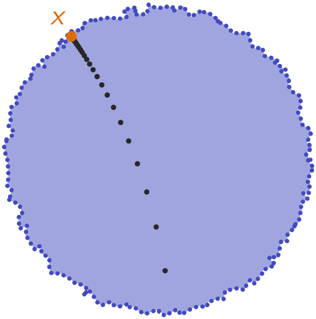
What are radial spaces?

Definition

X is *Fréchet-Urysohn* at x if whenever $x \in \overline{A}$, there exists a sequence $(x_n)_{n<\omega}$ in A that converges to x.
If X is Fréchet-Urysohn at every point x in X, then we say that the space is *Fréchet-Urysohn*.

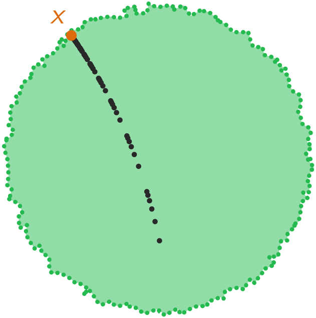


Fréchet-Urysohn space
$(x_n)_{n<\omega} \to x$

What are radial spaces?

Definition

X is *radial* at x if whenever $x \in \overline{A}$, there exists a *transfinite* sequence $(x_\alpha)_{\alpha<\lambda}$ in A that converges to x.
If X is radial at every point x in X, then we say that the space is *radial*.



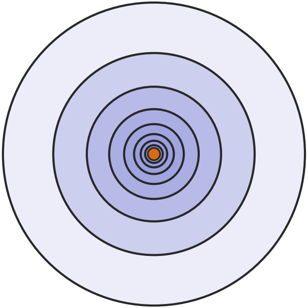
Radial space

$(x_\alpha)_{\alpha<\lambda} \to x$

Some examples

Definition

X is *first-countable* at x if there exists a countable neighbourhood base for x.
Equivalently, there exists a descending neighbourhood base $(U_n)_{n<\omega}$ for x.
If X is first-countable at every point x in X, then we say that the space is *first-countable*.

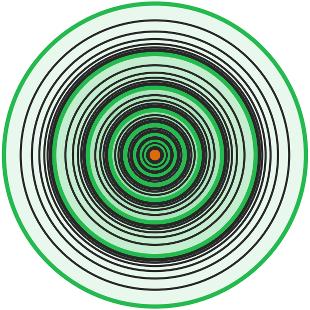


First countable space

Some examples

Definition

X is *well-based* at x if it has a well-ordered neighbourhood base with respect to $\supseteq$.
If X is well-based at every point x in X, then way say that the space is *well-based*.

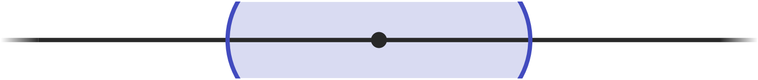


Well-based space

Some examples

Definition

LOTS := Linearly-Ordered Topological Space
GO-space := Generalised-Ordered space = Subspaces of LOTS



LOTS (or GO-space)

Some examples

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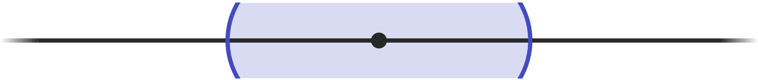

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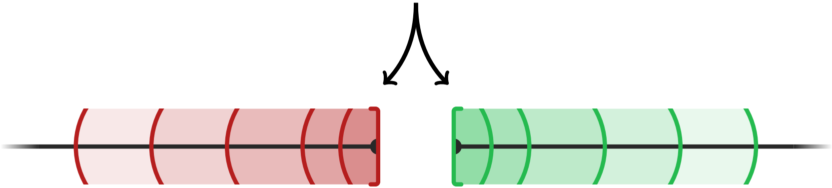

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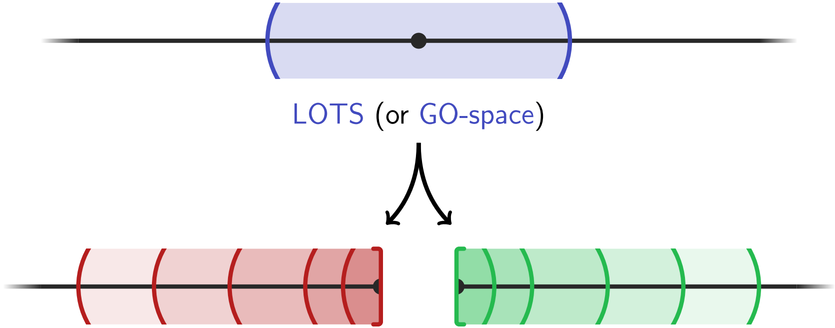

Definition

A *spoke* for a point is a well-based subspace containing that point.

Introducing spoke systems

Definition

A collection of spokes $\mathscr{S}$ for a point x is a *spoke system* for x if

$$\mathscr{B} := \left\{ \bigcup_{S \in \mathscr{S}} B_S : \forall S \in \mathscr{S}, B_S \in \mathscr{N}_x^S \right\}$$

is a neighbourhood base for x, where $\mathscr{N}_x^S$ is the collection of S-neighbourhoods of x, for each $S \in \mathscr{S}$.

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Theorem

Every point with a spoke system is radial.

Introducing spoke systems

Definition

A transfinite sequence $(x_\alpha)_{\alpha<\lambda}$ *converges strictly* to a point x if it converges to x and x is not in the closure of any initial segment; that is, $x \notin \overline{\{x_\alpha : \alpha < \beta\}}$, for all $\beta < \lambda$.

Lemma

If X is radial at x and $x \in \overline{A}$, then there exists an injective transfinite sequence in A that converges strictly to x.

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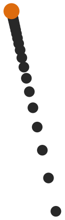
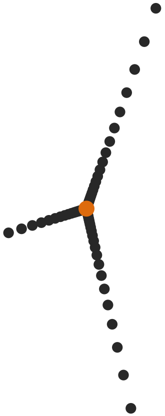
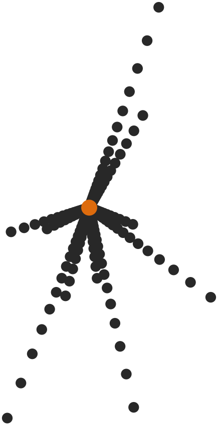
If X is radial at x and $x \in \overline{A}$, then there exists an injective transfinite sequence in A that converges strictly to x.

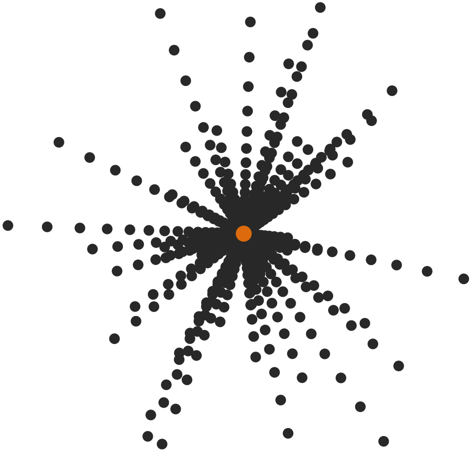
Lemma

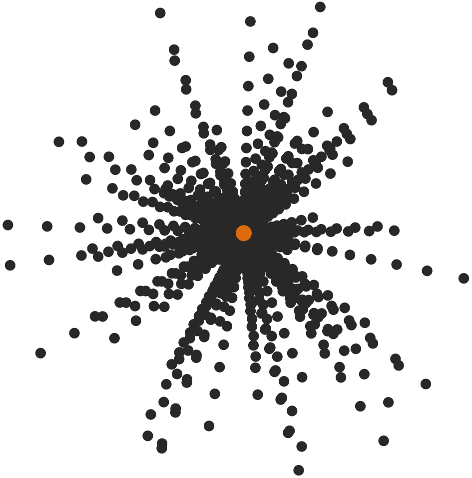
Let $(x_\alpha)_{\alpha<\lambda}$ be an injective transfinite sequence that converges strictly to x. Then $S_{(x_\alpha)_{\alpha<\lambda}} := \{x\} \cup \{x_\alpha : \alpha < \lambda\}$ is a spoke for x.

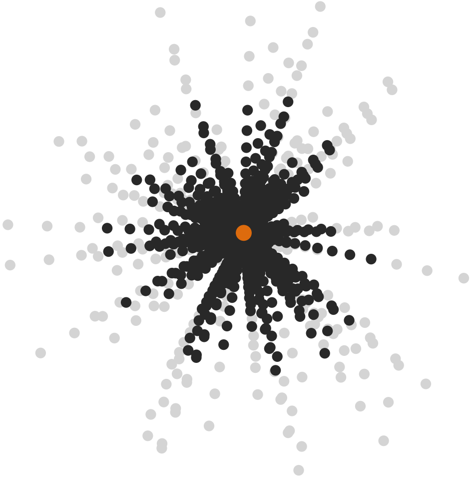












An internal characterisation of radiality

Theorem

For a point x in a topological space X, the following are equivalent:

1. *X is radial at x.*
2. *X has an almost-independent spoke system $\mathscr{S}$ at x; that is, for distinct $S, T \in \mathscr{S}, x \notin \overline{(S \cap T)\setminus\{x\}}$.*

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Proof.

If X is radial at x and not isolated, define:

$$\mathscr{T} := \{f : \lambda \to X \setminus \{x\} \mid \lambda \leq |X|, f \text{ is injective and } f \to x \text{ strictly}\}$$
$$\mathscr{A} := \{\mathscr{F} \subseteq \mathscr{T} : \forall f, g \in \mathscr{F} \text{ distinct}, f^{-1}[\mathrm{ran}(g)] \text{ is bdd. in } \mathrm{dom}(f)\}$$

Pick $\mathscr{F} \in \mathscr{A}$ maximal and define $\mathscr{S} := \{S_f : f \in \mathscr{F}\}$. □

Some properties of spoke systems

Proposition

Let $\mathscr{S}$ be a spoke system for x and $(x_\alpha)_{\alpha<\lambda}$ be a transfinite sequence clustering at x with $x \notin \overline{\{x_\alpha : \alpha < \beta\}}$ for $\beta < \lambda$, where λ is a regular ordinal. Then there exists an $S \in \mathscr{S}$ and a subsequence of $(x_\alpha)_{\alpha<\lambda}$ contained in S and converging to x.

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Proof.

As $x \in \overline{\{x_\alpha : \alpha < \lambda\}}$, there exists an $S \in \mathscr{S}$ such that $x \in \overline{\{x_\alpha : \alpha < \lambda\} \cap S}$. Then $\chi(x, S) = \lambda \ldots$ □

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Proposition

If $\mathscr{S}$ is an independent spoke system for x and $(x_\alpha)_{\alpha<\lambda} \subseteq X \setminus \{x\}$ converges to x, with λ regular, then there exists $\mathscr{T} \in [\mathscr{S}]^{<\lambda}$ and $\beta < \lambda$ such that $\{x_\alpha : \alpha \in [\beta, \lambda)\} \subseteq \bigcup \mathscr{T}$.

Strongly Fréchet spaces

Definition (Strongly Fréchet)

A point x in a space X is *strongly Fréchet* if for every decreasing sequence of subsets (A_n) with $x \in \bigcap_{n\in\omega} \overline{A_n}$, there exists a sequence (x_n) converging to x with $x_n \in A_n$ for all $n \in \omega$.

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Theorem

Let x be a Fréchet-Urysohn point in X. Then x is strongly Fréchet if and only if for all (non-trivial) spoke systems $\mathscr{S}$ at x and every countably infinite subset $\mathscr{A} \subseteq \mathscr{S}$, there exists an $S \in \mathscr{S}$ such that $A \cap S \neq \{x\}$ for all $A \in \mathscr{A}$.

Sketch proof.

$\Rightarrow$: consider $A_n := \bigcup_{m=n}^{\infty} S_m$, where $(S_m) \subseteq \mathscr{S}$.
$\Leftarrow$: use Zorn's Lemma (similar to proof of existence of almost-independent spoke systems). □

Independently-based spaces

Definition (Independently-based)

We say that a point x is *independently-based* if it has an *independent* spoke system $\mathscr{S}$; that is, $S \cap T = \{x\}$ for all distinct $S, T \in \mathscr{S}$.

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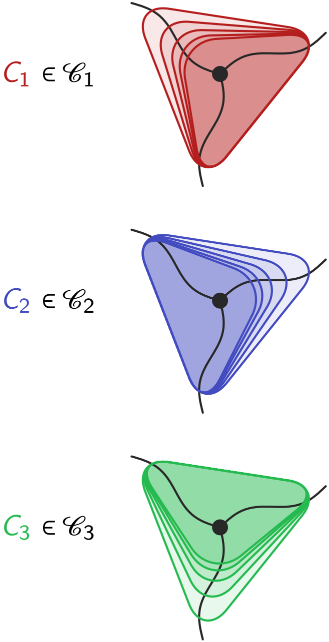
We say that a point x is *independently-based* if it has an *independent* spoke system $\mathscr{S}$; that is, $S \cap T = \{x\}$ for all distinct $S, T \in \mathscr{S}$. Equivalently, there exists a collection $\mathscr{C}$ of nests of neighbourhoods of x such that

$$\left\{ \bigcap_{C \in \mathscr{C}} U_C : \forall C \in \mathscr{C}, U_C \in \mathscr{C} \right\}$$

is a neighbourhood base for x and for every selection $(U_C)_{C \in \mathscr{C}}$,

$$\bigcap_{C \in \mathscr{C}} U_C = \bigcup_{C \in \mathscr{C}} (U_C \cap S_C)$$

where $S_C := \bigcap \{ \bigcap D : D \in \mathscr{C}, D \neq C \}$.



$C_1 \in \mathscr{C}_1$
$C_2 \in \mathscr{C}_2$
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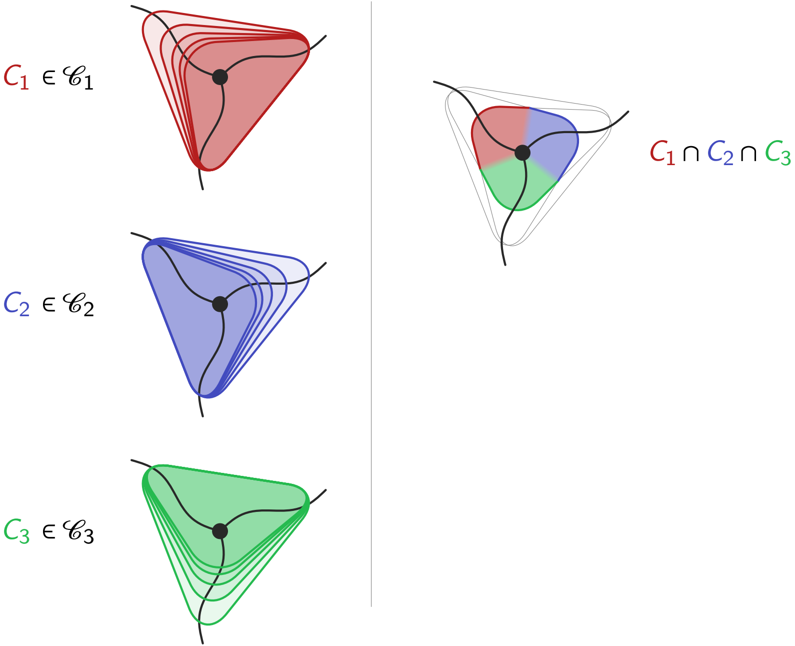

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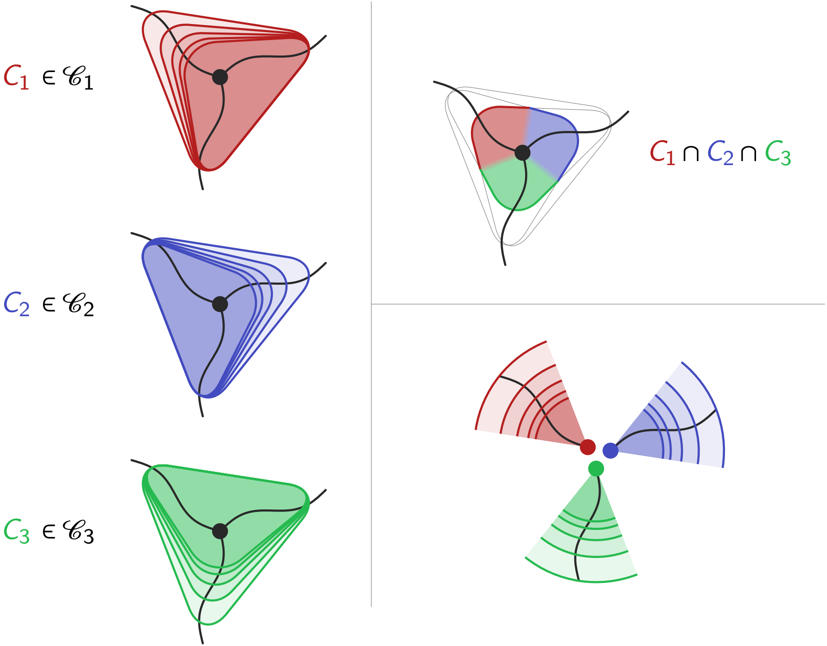
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Theorem

A point x in a space X is first countable if and only if it is independently-based and strongly Fréchet.

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Corollary

There exists a Fréchet-Urysohn space that is not independently-based.

Proof.

Take $X = \alpha D(\aleph_1)$. $\square$

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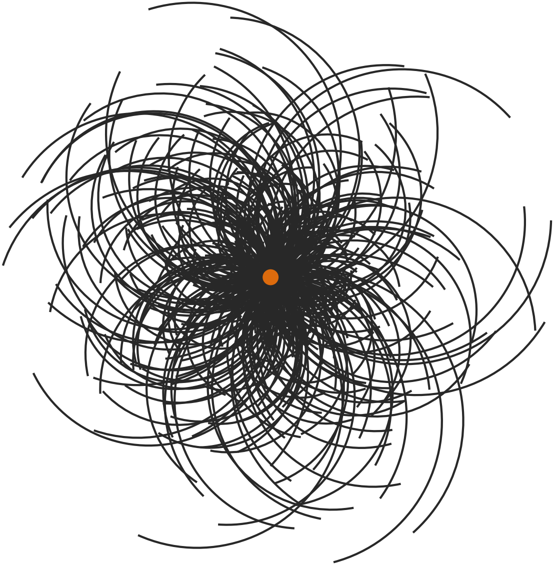
Theorem

There exists a Fréchet-Urysohn space with a point that is neither strongly Fréchet nor independently-based.



Lemma (Reflection Lemma)

Let x be a Fréchet-Urysohn point, $\mathscr{S}, \mathscr{T}$ be spoke systems at x, with $\mathscr{T}$ independent. Then for all $K_S := \{T \in \mathscr{T} : x \in \overline{(S \cap T) \setminus \{x\}}\}$ is finite, for all $S \in \mathscr{S}$.

Independently-based spaces

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Sketch proof of previous theorem.

For $x \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, define $S_x := \{y \in \mathbb{R}^2 : \|y - x\| = \|x\|\}$ and let $\mathscr{B} := \{\bigcup_{x \in \mathbb{R}^2 \setminus \{\mathbf{0}\}} (S_x \cap B(\mathbf{0}, \varepsilon_x)) : \forall x \in \mathbb{R}^2 \setminus \{\mathbf{0}\}, \varepsilon_x > 0\}$. If $\mathbf{0}$ is independently-based, then for each $x \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, there exists an $\varepsilon_x > 0$ such that $S_x \cap S_y \cap B(\mathbf{0}, \min(\varepsilon_x, \varepsilon_y)) = \{\mathbf{0}\}$ for all distinct $x, y \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$. By the Baire category theorem, we obtain a contradiction. □

Some problems

Proposition

If x is a Fréchet-Urysohn, non-first-countable point with a countable, almost-independent spoke system, then $\chi(x,X) = \mathfrak{d}$.

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Question

What if x has no countable, almost-independent spoke system?

Some problems

Proposition

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Question

What if x has no countable, almost-independent spoke system?

Question

Let $\mathscr{A}$ be an almost-disjoint family on ω and topologise $\omega \cup \{\star\}$ by declaring A to be a sequence converging to $\star$ and $\{A \cup \{\star\} : A \in \mathscr{A}\}$ is a spoke system at $\star$ (so $\omega \cup \{\star\} \cong \Psi(\mathscr{A})/\mathscr{A}$). What is the character of $\star$?

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