Avoidable Polynomials and $\mathbb{R} \subseteq L$

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Törnquist and Weiss idea

In 2012 Törnquist and Weiss studied many Σ^1_2 definable version of some statements equivalent to CH ($2^{\aleph_0} = \aleph_1$).

CH $\iff$ there exist some **objects** such that **something happens**.

They proved that these Σ^1_2 counterparts become equivalent to the statement "all reals are constructible".

$\mathbb{R} \subseteq L \iff$ there exist some Σ^1_2 **objects** such that **something happens**.

From "CH implies S" to "$\mathbb{R} \subseteq L$ implies the Σ^1_2 version of S"

A Δ^1_2 well-ordering $\prec$ is **strong** if it has length ω_1 and if for any $P \subseteq \mathbb{R} \times \mathbb{R}$ which is Σ^1_2,

$$\forall z \prec y \; P(x, z)$$

is Σ^1_2 as well.

Theorem (Addison 1959)

If $\mathbb{R} \subseteq L$ then there exists a Δ^1_2 strong well-ordering of the reals.

From "S implies CH" to "the Σ^1_2 version of S implies $\mathbb{R} \subseteq L$"

Theorem(Mansfield and Solovay 1970)

Let A be a $\Sigma^1_2(a)$ set. Then either $A \subseteq L[a]$, or else A contains a perfect set. In particular, if a Σ^1_2 set contains a non-constructible real then it is uncountable.

Lemma (Törnquist and Weiss 2012)

1. If there exists a non-constructible real, there exists a non-constructible real $x \in V$ such that $\aleph_1^{L[x]} = \aleph_1^L$.
2. Let $a \in L$ and A be a $\Sigma^1_2(a)$ definable set. Then if A is uncountable, $A \cap L$ is uncountable in L.

Törnquist and Weiss results

Theorem (Sierpinski 1965)

CH holds iff there are two sets $A, B \subseteq \mathbb{R}^2$ with $A \cup B = \mathbb{R}^2$ such that all vertical sections of A are countable and all horizontal sections of B are countable.

Theorem (Törnquist and Weiss 2012)

$\mathbb{R} \subseteq L$ iff there are Σ^1_2 sets $A, B \subseteq \mathbb{R}^2$ with $A \cup B = \mathbb{R}^2$ such that all vertical sections of A are countable and all horizontal sections of B are countable.

Törnquist and Weiss results

Theorem (Sierpinski 1965)

CH holds iff there are sets $A_1, A_2, A_3 \subseteq \mathbb{R}^3$ such that $A_1 \cup A_2 \cup A_3 = \mathbb{R}^3$, and every line in the direction of the x_i-axis meets A_i in finitely many points.

Theorem (Törnquist and Weiss 2012)

$\mathbb{R} \subseteq L$ holds iff there are Σ^1_2 sets $A_1, A_2, A_3 \subseteq \mathbb{R}^3$ such that $A_1 \cup A_2 \cup A_3 = \mathbb{R}^3$, and every line in the direction of the x_i-axis meets A_i in finitely many points.

Törnquist and Weiss results

Theorem (Komjáth and Totik 2006)

$\neg$CH implies that for any $n \in \omega$ and any $f : \mathbb{R} \times \mathbb{R} \to \omega$ there exist two sets $C, D \subseteq \mathbb{R}$ such that $|C| = |D| = n$ and $f \restriction C \times D$ is monochromatic.

Theorem (Törnquist and Weiss 2012)

$\mathbb{R} \not\subseteq L$ iff for any $n \in \omega$ and for every Σ^1_2-definable function $f : \mathbb{R} \times \mathbb{R} \to \omega$ there are sets $C, D \subseteq \mathbb{R}$ such that $|C| = |D| = n$ and $f \restriction C \times D$ is monochromatic.

Törnquist and Weiss results

Theorem (Komjáth and Totik 2006)

$\neg$CH implies that for any coloring $g : \mathbb{R} \to \omega$ there are four distinct $x, y, z, w \in \mathbb{R}$ of the same color such that

$$x + y = z + w.$$

Theorem (Törnquist and Weiss 2012)

$\mathbb{R} \not\subseteq L$ iff for any Σ^1_2 coloring $g : \mathbb{R} \to \omega$ there are four distinct $x, y, z, w \in \mathbb{R}$ of the same color such that

$$x + y = z + w.$$

Some algebraic equivalences

Theorem (Erdős and Kakutani 1943)

CH is equivalent to the following proposition: the set of all real numbers can be decomposed into a countable number of subsets, each consisting only of rationally independent numbers.

Proposition

$\mathbb{R} \subseteq L$ iff there exists $\psi(x, i)$ Σ^1_2 such that $x \in S_i \iff \psi(x, i)$ and $\mathbb{R} = \bigcup\{S_i : i \in \omega\}$ where each S_i consists only of rationally independent numbers.

Some algebraic equivalences

Theorem (Zoli 2006)

CH holds if and only if the set of all transcendental reals is a union of countably many transcendence bases for $\mathbb{R}$.

Proposition

$\mathbb{R} \subseteq L$ iff the set of all transcendental reals is the union of countably many transcendence bases for $\mathbb{R}$ uniformly defined by a Σ^1_2 predicate.

(k, n)-ary polynomials

A polynomial $p(x_0, \ldots, x_{k-1}) \in \mathbb{R}[x_0, \ldots, x_{k-1}]$ is a **(k, n)-ary polynomial** if every x_i is an n-tuple of variables.

For instance

$$p(x, y, z) = \| x - y \|^2 - \| y - z \|^2 .$$

is a $(3, n)$-ary polynomial. Note that the product is the **scalar product**.

Avoidable (k, n)-ary polynomials

- Given a (k, n)-ary polynomial $p(x_0, \ldots, x_{k-1})$, a coloring

$$\chi : \mathbb{R}^n \to \omega$$

avoids $p(x_0, \ldots, x_{k-1})$ if for every $r_0, \ldots, r_{k-1} \in \mathbb{R}^n$ distinct and monochromatic with respect to χ,

$$p(r_0, \ldots, r_{k-1}) \neq 0.$$

- The polynomial $p(x_0, \ldots, x_{k-1})$ is **avoidable** if there exists a coloring which avoids it.

- A function

$$\alpha : A_0 \times A_1 \times \cdots \times A_{m-1} \to B_0 \times B_1 \times \cdots \times B_{m-1}$$

is **coordinately induced** if for every $i \in m$ there is a function $\alpha_i : A_i \to B_i$ such that

$$\alpha(a_0, \ldots, a_{m-1}) = (\alpha_0(a_0), \ldots, \alpha_{m-1}(a_{m-1})).$$

- A function

$$g : A^m \to B$$

is **one-one in each coordinate** if for every $a_0, \ldots, a_{m-1} \in A$ and $b \in A$, $b \neq a_i$ for some $i \in m$, then

$$g(a_0, \ldots a_{i-1}, a_i, a_{i+1}, \ldots, a_{m-1}) \neq g(a_0, \ldots, a_{i-1}, b, a_{i+1}, \ldots a_{m-1}).$$

Schmerl's definition of m-avoidance

Let $n \in \omega$ and $k \in \omega \setminus \{0,1\}$. For each $m \in \omega$ we say that a (k,n)-ary polynomial $p(x_0, \ldots, x_{k-1})$ is ***m*-avoidable** if for each definable function

$$g : (0,1)^m \to \mathbb{R}^n$$

which is one-one in each coordinate and for distinct

$$e_0, \ldots, e_{k-1} \in (0,1)^m$$

there is a coordinately induced

$$\alpha : (0,1)^m \to (0,1)^m$$

such that

$$p(g\alpha(e_0), \ldots, g\alpha(e_{k-1})) \neq 0.$$

The relationship between avoidance and m-avoidance

Theorem (Schmerl 1999)

If $\neg$CH holds then every avoidable polynomial is 2-avoidable.

Theorem (Schmerl 1999)

If CH holds then every 1-avoidable polynomial is avoidable.

Σ^1_2 avoidance

- A (k, n)-ary polynomial $p(x_0, \ldots, x_{k-1})$ is **Σ^1_2-avoidable** if there exists a Σ^1_2 coloring which avoids it.
- A (k, n)-ary polynomial $p(x_0, \ldots, x_{k-1})$ is **(m, Σ^1_2)-avoidable** if for each $r \in \mathbb{R} \cap L$ and for each $\Sigma^1_2(r)$ function

  $$g : (0, 1)^m \to \mathbb{R}^n$$

  which is one-one in each coordinate and for distinct

  $$e_0, \ldots, e_{k-1} \in (0, 1)^m$$

  there is a coordinately induced

  $$\alpha : (0, 1)^m \to (0, 1)^m$$

  which is $\Sigma^1_2(r, e_0, \ldots, e_{k-1})$ and such that

  $$p(g\alpha(e_0), \ldots, g\alpha(e_{k-1})) \neq 0.$$

Σ^1_2 versions of Schmerl's results

Theorem (Schmerl 1999)

If $\neg$CH holds then every avoidable polynomial is 2-avoidable.

Proposition

If $\mathbb{R} \not\subseteq L$ then every Σ^1_2-avoidable polynomial is $(2, \Sigma^1_2)$-avoidable.

Σ^1_2 versions of Schmerl's results

Theorem (Schmerl 1999)

If CH holds then every 1-avoidable polynomial is avoidable.

Proposition

If $\mathbb{R} \subseteq L$ then every $(1, \Sigma^1_2)$-avoidable polynomial is Σ^1_2-avoidable.

Erdős and Komjáth equivalence

Theorem (Erdős and Komjáth 1990)

CH holds if and only if the plane can be colored with countably many colors with no monochromatic right-angled triangle.

Proposition

$\mathbb{R} \subseteq L$ if and only if there exists a Σ^1_2 coloring of the plane with countably many colors with no monochromatic right-angled triangle.

Why is it a corollary of the Σ^1_2 version of Schmerl's result?

Proposition

$\mathbb{R} \subseteq L$ if and only if there exists a Σ^1_2 coloring of the plane with countably many colors with no monochromatic right-angled triangle.

Since it happens iff the $(3,2)$-polynomial:

$$p(x,y,z) = \| x - y \|^2 + \| z - y \|^2 - \| x - z \|^2$$

is Σ^1_2-avoidable.

Why is it a corollary of the Σ^1_2 version of Schmerl's result?

Proposition

$\mathbb{R} \subseteq L$ if and only if there exists a Σ^1_2 coloring of the plane with countably many colors with no monochromatic right-angled triangle.

Since it happens iff the $(3,2)$-polynomial:

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Thank you!